

Electrical Engineering 1

12026105

Lecture 8

Second-Order Circuits

Second-Order Circuits Chapter 8

8.1 Examples of 2nd order RCL circuit

8.2 The source-free series RLC circuit

8.3 The source-free parallel RLC circuit

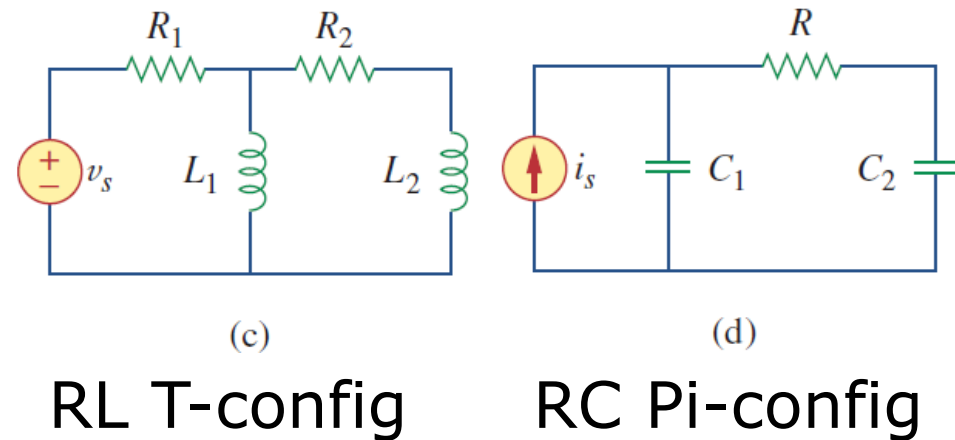
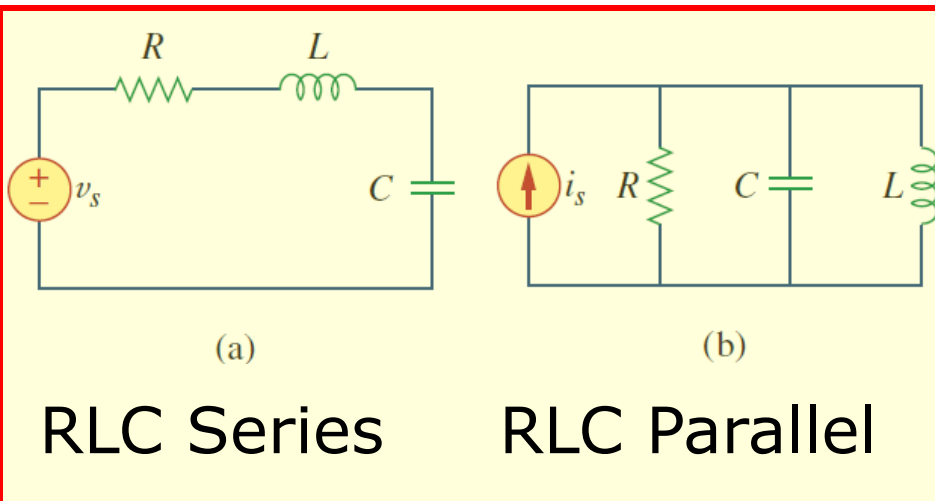
8.4 Step response of a series RLC circuit

8.5 Step response of a parallel

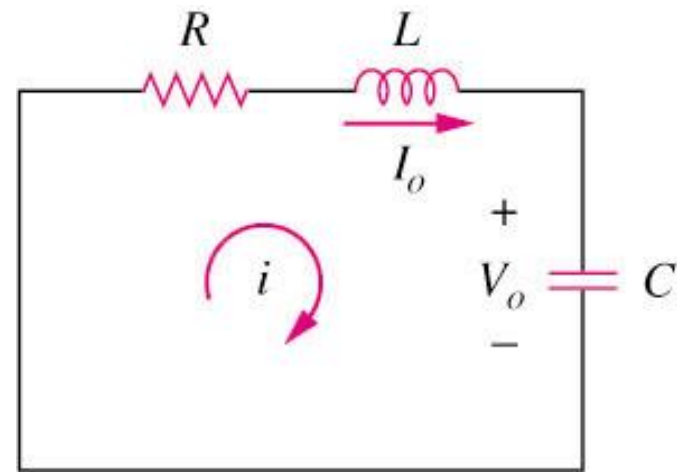
8.1 Second Order RLC circuits (1)

What is a 2nd order circuit?

A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.



8.2 Source-Free Series RLC Circuits



➤ The solution of the source-free series RLC circuit is called as the natural response of the circuit.

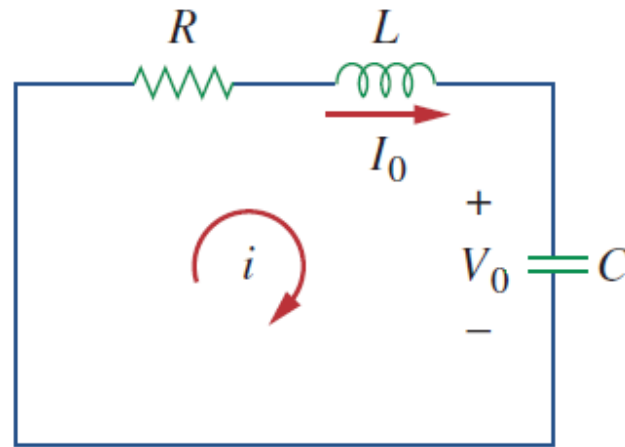
➤ The circuit is excited by the energy initially stored in the capacitor and inductor.

The 2nd order
of expression

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

How to derive and how to solve?

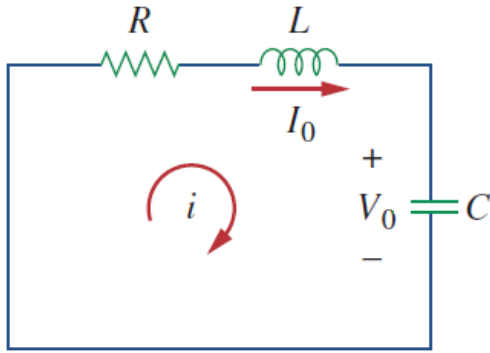
8.2 Source-Free Series RLC Circuits



For Capacitor : $v(0) = v(0^+) = v(0^-) = V_0$

For Inductor : $i(0) = i(0^+) = i(0^-) = I_0$

8.2 Source-Free Series RLC Circuits



Initial Conditions

$$i(0) = I_0$$

$$v(0) = \frac{1}{C} \int_{-\infty}^0 i dt = V_0$$

Apply KVL

$$Ri + L \frac{di}{dt} + \frac{1}{C} \int_{-\infty}^t i dt = 0$$

$$R \frac{di}{dt} + L \frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

To solve such a 2nd order diff eq. We need 2 initial conditions, such as $i(0)$ and $\frac{di(0)}{dt}$ (from $v(0)$)

8.2 Source-Free Series RLC Circuits

We get the initial value of the derivative of I from equation after applying KVL; that is,

$$Ri(0) + L \frac{di(0)}{dt} + v(0) = 0$$

$$RI_0 + L \frac{di(0)}{dt} + V_0 = 0$$

2 Initial Conditions

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

$$i(0) = I_0$$

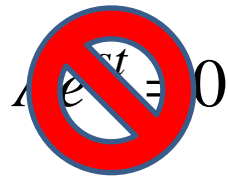
8.2 Source-Free Series RLC Circuits

In 1st order circuits suggests that the solution is $I_0 e^{-t/\tau}$. So we let

$i = Ae^{st}$ Where A and s have to be determined

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

$$As^2 e^{st} + \frac{AR}{L} s e^{st} + \frac{Ae^{st}}{LC} = 0 \quad \Rightarrow \quad Ae^{st} \left(s^2 + \frac{R}{L} s + \frac{1}{LC} \right) = 0$$

$$Ae^{st} = 0$$


$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

$$s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

8.2 Source-Free Series RLC Circuits

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s^2 + \alpha s + \omega_0^2 = 0$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

8.2 Source-Free Series RLC Circuits

There are three possible solutions for the following 2nd order differential equation:

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = 0$$

General 2nd order Form

$$\text{Where } \alpha = \frac{R}{2L} \text{ and } \omega_0 = \sqrt{\frac{1}{LC}}$$

The types of solutions for i depend on the relative values of α and ω_0

8.2 Source-Free Series RLC Circuits

There are three possible solutions for the following 2nd

order differential equation: $\frac{d^2i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = 0$

1. If $\alpha > \omega_0$, over-damped case

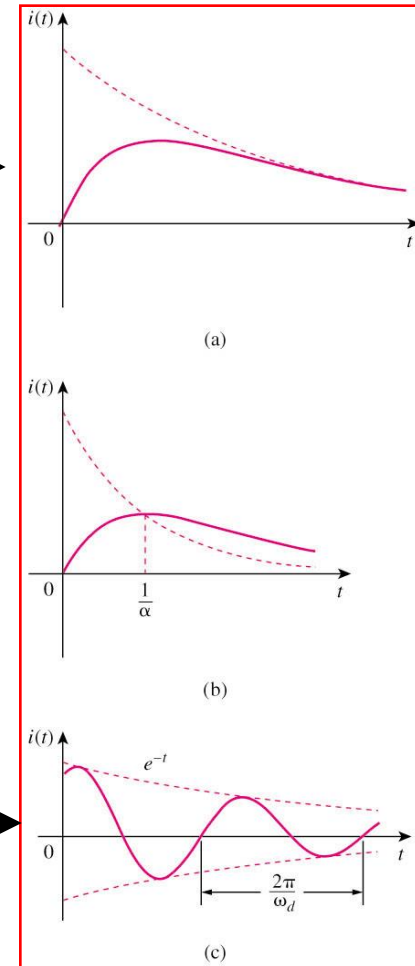
$$i(t) = Ae^{s_1 t} + Be^{s_2 t} \quad \text{where} \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

2. If $\alpha = \omega_0$, critical damped case

$$i(t) = (A + Bt)e^{-\alpha t} \quad \text{where} \quad s_{1,2} = -\alpha$$

3. If $\alpha < \omega_0$, under-damped case

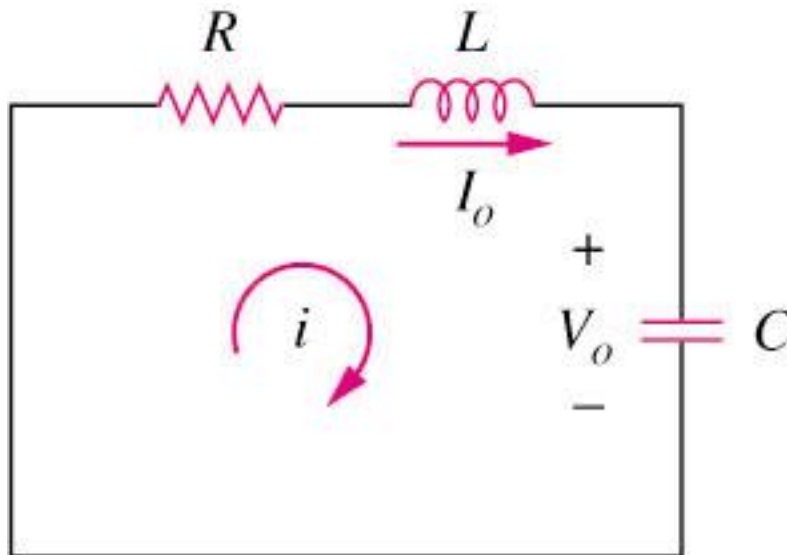
$$i(t) = e^{-\alpha t} (A \cos \omega_d t + B \sin \omega_d t) \quad \text{where} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$



8.2 Source-Free Series RLC Circuits

Example 1

If $R = 10\Omega$, $L = 5\text{H}$, and $C = 2\text{mF}$ in figure below, find α , ω_0 , s_1 and s_2 . What type of natural response will the circuit have?

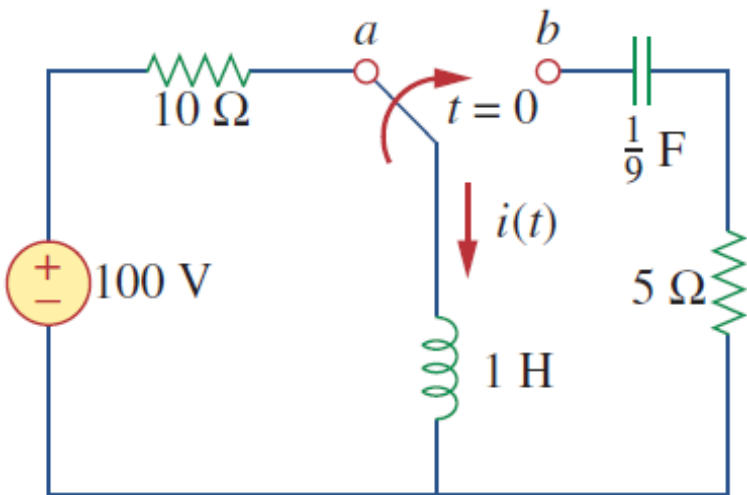


$$\alpha = \frac{R}{2L} = \frac{10}{2(5)} = 1$$
$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{5(0.002)}} = 10$$

Answer: *underdamped*

8.2 Source-Free Series RLC Circuits

Example 2 The circuit shown below has reached steady state at $t=0^-$. If the make-before-break switch moves to position b at $t=0$, calculate $i(t)$ for $t > 0$.



$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) = -\frac{1}{1}(50 + 0) = -50$$

$$i(0) = I_0 = 10\text{A}$$

$$\alpha = \frac{R}{2L} = \frac{5}{2(1)} = 2.5$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1(9)}{1}} = 3$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{3^2 - 2.5^2} = 1.6583$$

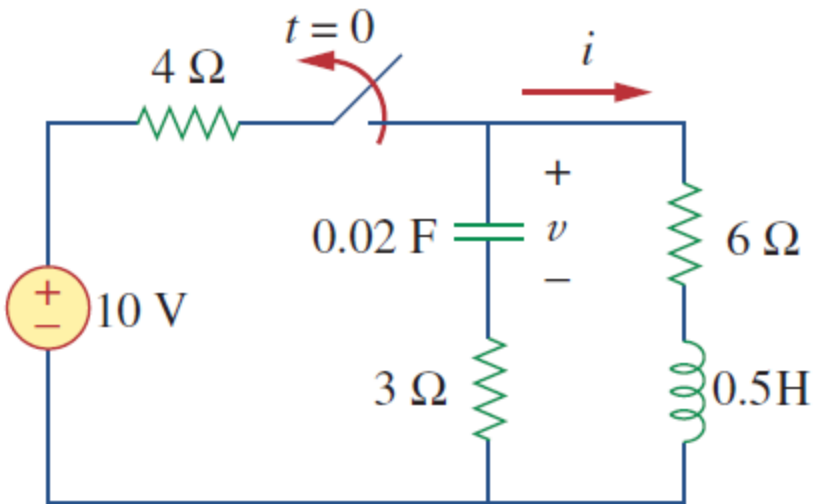
underdamped

$$i(t) = e^{-\alpha t} (A \cos \omega_d t + B \sin \omega_d t) = e^{-2.5t} (10 \cos(1.6583t) - 15.076 \sin(1.6583t))\text{A}$$

Answer: $i(t) = e^{-2.5t} (10 \cos(1.6583t) - 15.076 \sin(1.6583t))\text{A}$

8.2 Source-Free Series RLC Circuits

Example 3 Find $i(t)$ in the circuit of Figure below. Assume that the circuit has reached steady state at $t=0^-$.



$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) = -\frac{1}{0.5}(9 - 6) = -6$$

$$i(0) = I_0 = 1\text{A}$$

$$\alpha = \frac{R}{2L} = \frac{9}{2(0.5)} = 9$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.5(0.02)}} = 10$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^2 - 9^2} = 4.3589$$

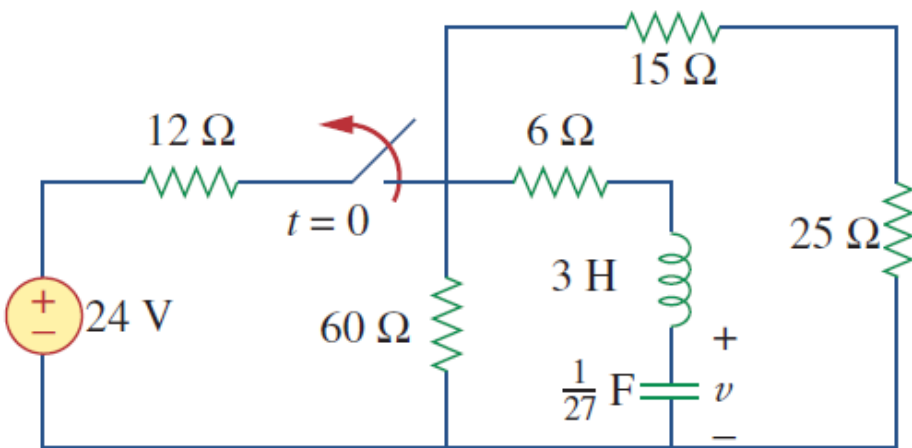
underdamped

$$i(t) = e^{-\alpha t} (A \cos \omega_d t + B \sin \omega_d t) = e^{-9t} (1 \cos(4.3589t) + 0.6882 \sin(4.3589t))\text{A}$$

Answer: $i(t) = e^{-9t} (\cos(4.3589t) + 0.6882 \sin(4.3589t))\text{A}$

8.2 Source-Free Series RLC Circuits

Example 4 Find $v(t)$ for $t > 0$



$$v(0) = V_0 = \left(\frac{24}{24+12} \right) 24 = 16\text{V} \quad i(0) = I_0 = 0$$

$$\alpha = \frac{R}{2L} = \frac{30}{2(3)} = 5$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{27}{3}} = 3$$

overdamped

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -5 + \sqrt{5^2 - 3^2} = -1$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -5 - \sqrt{5^2 - 3^2} = -9$$

$$v(t) = Ae^{-s_1 t} + Be^{-s_2 t} = Ae^{-t} + Be^{-9t}$$

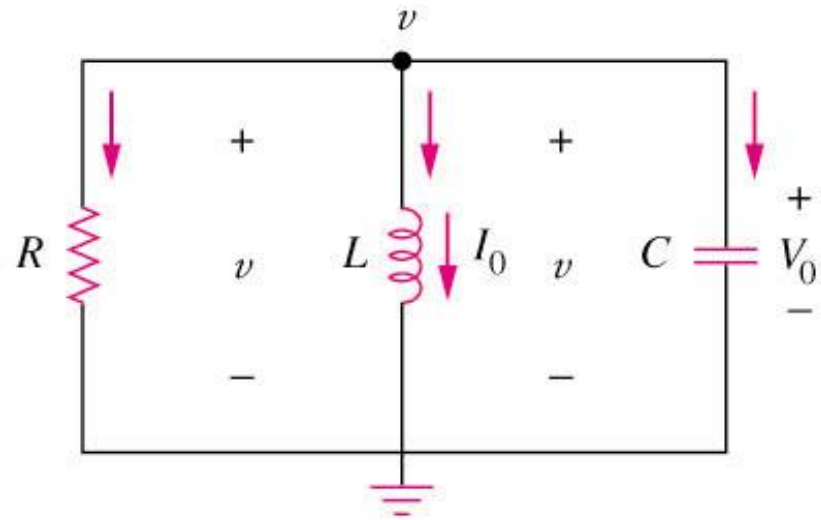
$$v(0) = 16 = A + B$$

$$i(t) = C \frac{dv}{dt} = \frac{1}{27} (-Ae^{-t} - 9Be^{-9t})$$

$$i(0) = 0 = \frac{1}{27} (-A - 9B) \rightarrow A = -9B \quad B = -2, A = 18$$

Answer: $v(t) = 18e^{-t} - 2e^{-9t}$

8.3 Source-Free Parallel RLC Circuits



Let
$$i(0) = I_0 = \frac{1}{L} \int_{-\infty}^0 v(t) dt$$

$v(0) = V_0$, Apply KCL to the top node:

$$\frac{v}{R} + \frac{1}{L} \int_{-\infty}^t v dt + C \frac{dv}{dt} = 0$$

Taking the derivative with respect to t and dividing by C

The 2nd order
of expression

$$\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = 0$$

8.3 Source-Free Parallel RLC Circuits

There are three possible solutions for the following 2nd order differential equation:

$$\frac{d^2v}{dt^2} + 2\alpha \frac{dv}{dt} + \omega_0^2 v = 0 \quad \text{where} \quad \alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

1. If $\alpha > \omega_0$, over-damped case

$$v(t) = A e^{s_1 t} + B e^{s_2 t} \quad \text{where} \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

2. If $\alpha = \omega_0$, critical damped case

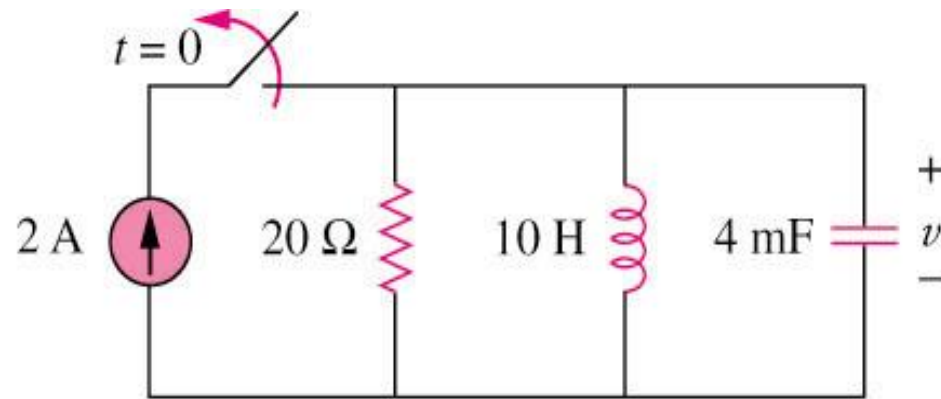
$$v(t) = (A + Bt) e^{-\alpha t} \quad \text{where} \quad s_{1,2} = -\alpha$$

3. If $\alpha < \omega_0$, under-damped case

$$v(t) = e^{-\alpha t} (A \cos \omega_d t + B \sin \omega_d t) \quad \text{where} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

8.3 Source-Free Parallel RLC Circuits

Example 5 Refer to the circuit shown below. Find $v(t)$ for $t > 0$.



$$v(0) = 0, i(0) = 2$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(20)(0.004)} = 6.25$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10(0.004)}} = 5$$

overdamped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{6.25^2 - 5^2} = -6.25 \pm 3.75 = -2.5, -10$$

$$v(t) = A e^{s_1 t} + B e^{s_2 t} = A e^{-2.5t} + B e^{-10t} \quad v(0) = 0 = A + B$$

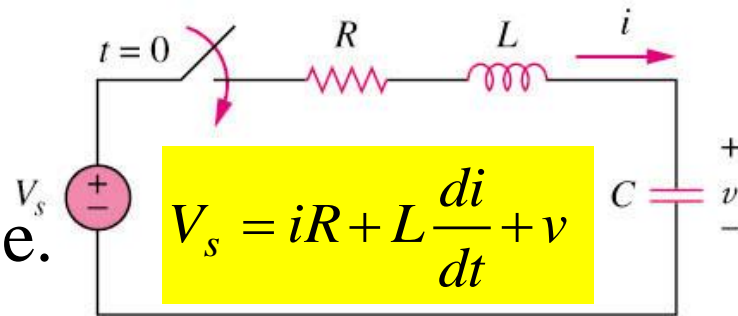
$$v(t) = L \frac{di}{dt} \rightarrow i = \frac{1}{L} \int v(t) dt \rightarrow i(t) = \frac{1}{10} \int (A e^{-2.5t} + B e^{-10t}) dt \quad i(0) = 2 = -\frac{A}{25} - \frac{B}{100}$$

$$B = \frac{200}{3}, A = -\frac{200}{3}$$

Answer: $v(t) = \frac{200}{3} (e^{-10t} - e^{-2.5t})$

8.4 Step-Response Series RLC Circuits

The step response is obtained by the sudden application of a dc source.



$$i = C \frac{dv}{dt}$$

$$V_s = RC \frac{dv}{dt} + LC \frac{d^2 v}{dt^2} + v$$

The 2nd order of expression $\frac{d^2 v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$

The above equation has the same form as the equation for source-free series RLC circuit.

- The same coefficients (important in determining the frequency parameters).
- Different circuit variable in the equation.

8.4 Step-Response Series RLC Circuits

The solution of the equation should have two components: the transient response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$v(t) = v_t(t) + v_{ss}(t)$$

- The transient response v_t is the same as that for source-free case

$$v_t(t) = Ae^{s_1 t} + Be^{s_2 t}$$

(over-damped)

$$v_t(t) = (A + Bt)e^{-\alpha t}$$

(critically damped)

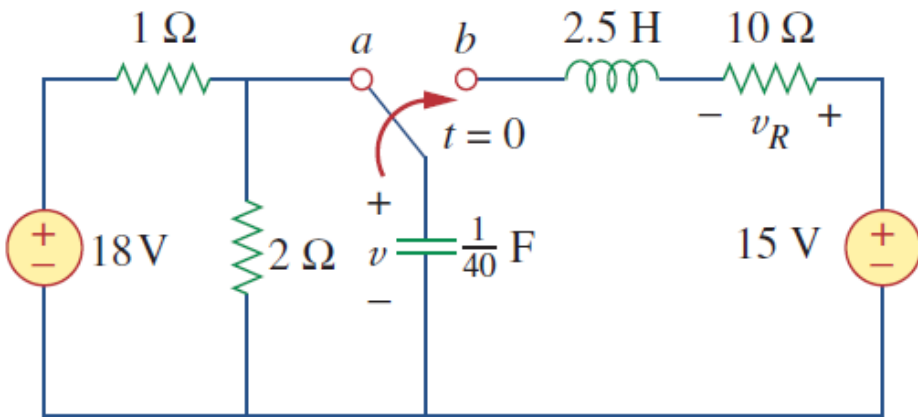
$$v_t(t) = e^{-\alpha t} (A \cos \omega_d t + B \sin \omega_d t)$$

(under-damped)

- The steady-state response is the final value of $v(t)$. $v_{ss}(t) = v(\infty)$
- A and B are obtained from the initial conditions : $v(0)$, $\frac{dv(0)}{dt}$

8.4 Step-Response Series RLC Circuits

Example 6 Having been in position for a long time, the switch in the circuit below is moved to position b at $t=0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.



$$v(0) = \frac{2}{3}(18) = 12V, \quad i(0) = 0$$

$$v_{ss}(t) = 15, \quad i(0) = C \frac{dv(0)}{dt} \rightarrow \frac{dv(0)}{dt} = 0$$

$$\alpha = \frac{R}{2L} = \frac{10}{2(2.5)} = 2, \quad \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{40}{2.5}} = 4$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 2\sqrt{3} \quad \text{under-damped}$$

$$v(t) = v_{ss}(t) + v_t(t) = 15 + e^{-2t} (A \cos 2\sqrt{3}t + B \sin 2\sqrt{3}t) \quad v(0) = 12 = 15 + A \rightarrow A = -3$$

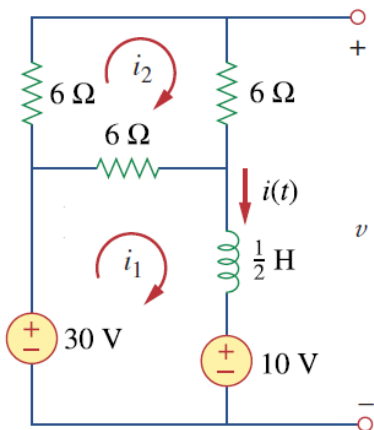
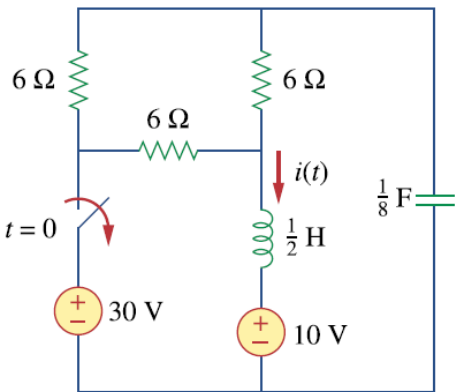
$$\frac{dv(t)}{dt} = \left(-2e^{-2t} (-3 \cos 2\sqrt{3}t + B \sin 2\sqrt{3}t) \right) + \left(e^{-2t} (6\sqrt{3} \sin 2\sqrt{3}t + 2\sqrt{3}B \cos 2\sqrt{3}t) \right)$$

$$\frac{dv(0)}{dt} = 0 = 6 + 2\sqrt{3}B \rightarrow B = -\sqrt{3}$$

Answer: $v(t) = 15 - e^{-2t} (3 \cos(2\sqrt{3}t) + \sqrt{3} \sin(2\sqrt{3}t)) V : v_R(t) = 2\sqrt{3} e^{-2t} \sin(2\sqrt{3}t) V$

8.4 Step-Response Series RLC Circuits

Example 7 Find $i(t)$ in the circuit of Figure below.



$$i(0) = I_0 = 5A$$

$$i_1 = 5A, i_2 = \frac{5}{3}A, v(0) = 20V$$

$$R = 12 // 6 = 4\Omega, \alpha = \frac{R}{2L} = \frac{4(2)}{2} = 4$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{16} = 4, \text{ critically-damped}$$

$$v(t) = v_{ss}(t) + v_t(t) = 10 + e^{-4t}(A + Bt)$$

$$v(0) = 20 = 10 + A \rightarrow A = 10$$

$$i(t) = C \frac{dv}{dt} = \frac{e^{-4t}}{8}(-4(A + Bt) + B)$$

$$i(0) = -5 = \frac{e^{-4t}}{8}(-4(A + Bt) + B)$$

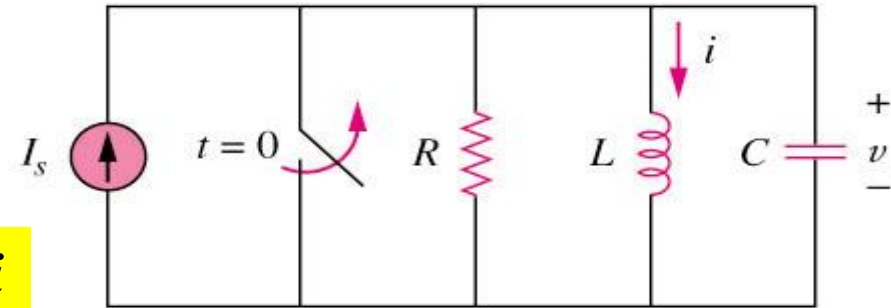
Answer: $i(t) = e^{-9t}(\cos(4.3589t) + 0.6882 \sin(4.3589t))A$

8.5 Step-Response Parallel RLC Circuits

The step response is obtained by the sudden application of a dc source.

$$C \frac{dv}{dt} + \frac{v}{R} + i = I_s$$

$$v = L \frac{di}{dt}$$



$$LC \frac{d^2 i}{dt^2} + \frac{L}{R} \frac{di}{dt} + i = I_s$$

The 2nd order of expression $\frac{d^2 i}{dt^2} + \frac{1}{RC} \frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$

It has the same form as the equation for source-free parallel RLC circuit.

- The same coefficients (important in determining the frequency parameters).
- Different circuit variable in the equation.

8.5 Step-Response Parallel RLC Circuits

The solution of the equation should have two components: the transient response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$i(t) = i_t(t) + i_{ss}(t)$$

- The transient response i_t is the same as that for source-free case

$$i_t(t) = Ae^{s_1 t} + Be^{s_2 t} \quad (\text{over-damped})$$

$$i_t(t) = (A + Bt)e^{-\alpha t} \quad (\text{critical damped})$$

$$i_t(t) = e^{-\alpha t} (A \cos \omega_d t + B \sin \omega_d t) \quad (\text{under-damped})$$

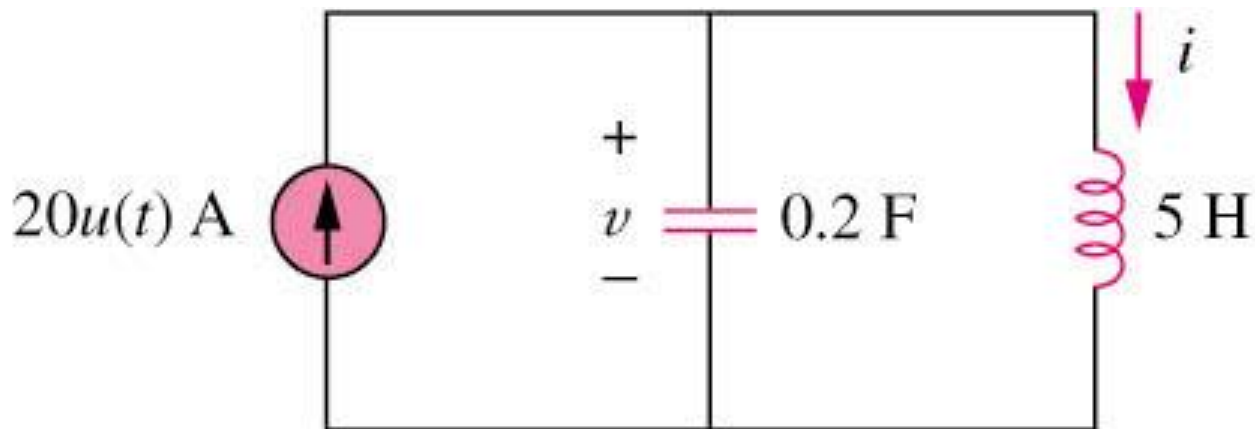
- The steady-state response is the final value of $i(t)$. $i_{ss}(t) = i(\infty) = I_s$

- The values of A and B are obtained from the initial conditions: $i(0), \frac{di(0)}{dt}$

8.5 Step-Response Parallel RLC Circuits

Example 8

Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit shown in circuit shown below:



Answer: $v(t) = Ldi/dt = 5 \times 20 \sin(t) = 100 \sin(t)$ V