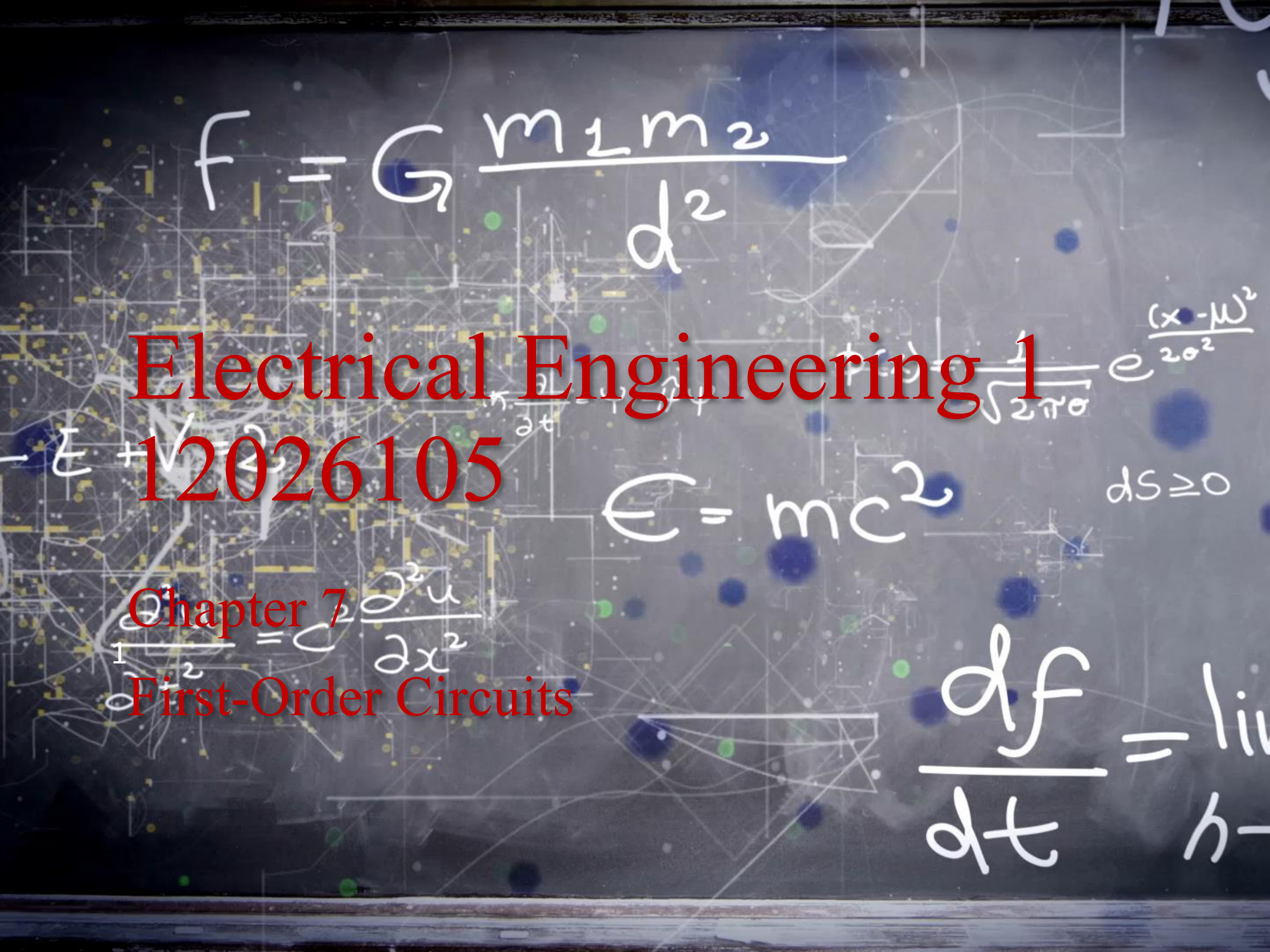


Electrical Engineering 1
12026105

Chapter 7

First-Order Circuits

Learning Objectives

By using the information and exercises in this chapter you will be able to:

1. Understand solutions to unforced, first-order linear differential equations.
2. Comprehend singularity equations and know their importance in solving linear differential equations.
3. Understand the effect of unit step sources on first-order linear differential equations.
4. Explain how dependent sources and op amps influence simple first-order linear differential equations.
5. Use *PSpice* to solve simple transient circuits with an inductor or a capacitor.

First-Order Circuits

7.1 The Source-Free RC Circuit (วงจร RC ที่ไม่มีแหล่งจ่าย)

7.2 The Source-Free RL Circuit (วงจร RL ที่ไม่มีแหล่งจ่าย)

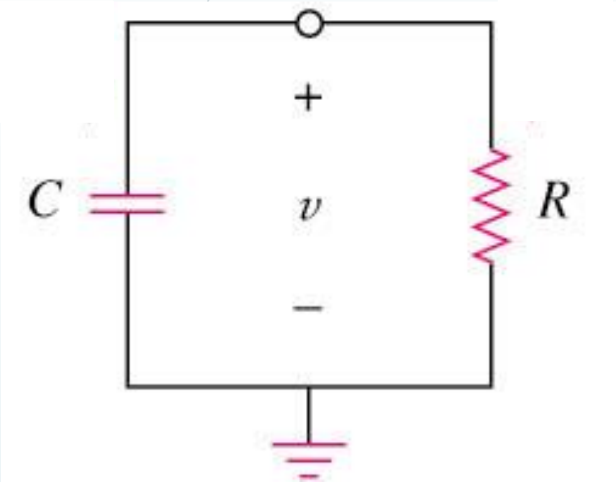
7.3 Singularity Functions (Switching Functions)

7.4 Step Response of an RC Circuit (วงจร RC ที่มีแหล่งจ่าย)

7.5 Step Response of an RL Circuit (วงจร RC ที่มีแหล่งจ่าย)

7.1 The Source-Free RC Circuit (1)

A first-order circuit is characterized by a first-order differential equation.



By taking KCL at top node

$$i_R + i_C = 0 \rightarrow \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law

Capacitor law

$$\frac{dv}{v} = -\frac{dt}{RC}$$

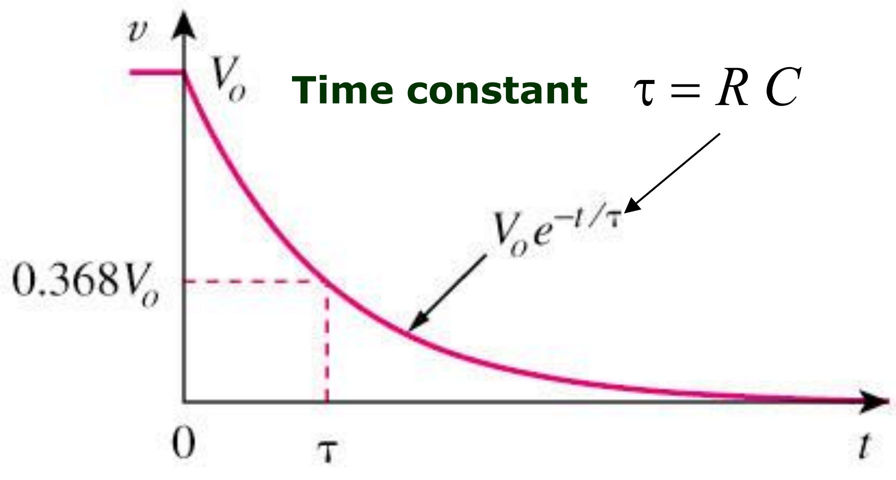
$$\ln(v) = -\frac{t}{RC} + Cont$$

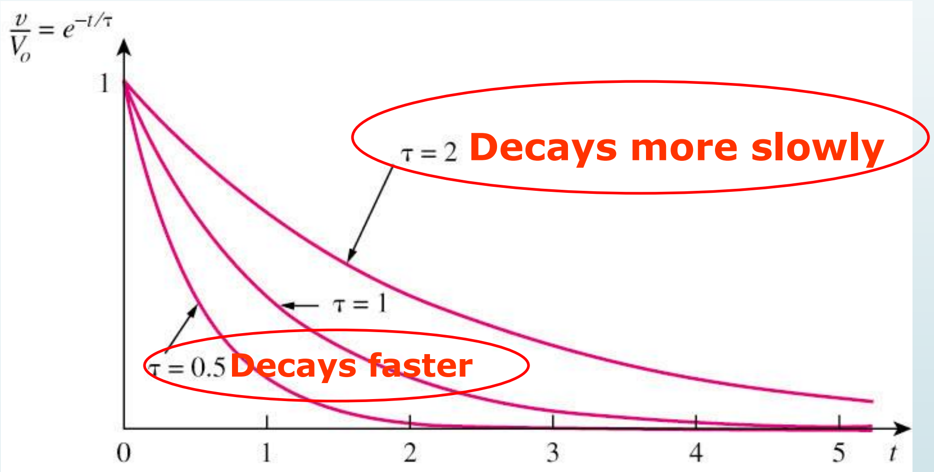
$$\boxed{v = e^{\left(-\frac{t}{RC}+Cont\right)} = V_o e^{\left(-\frac{t}{RC}\right)}}$$

- Apply Kirchhoff's laws to purely resistive circuit results in algebraic equations.
- Apply the laws to RC and RL circuits produces differential equations.

7.1 The Source-Free RC Circuit (2)

- The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.



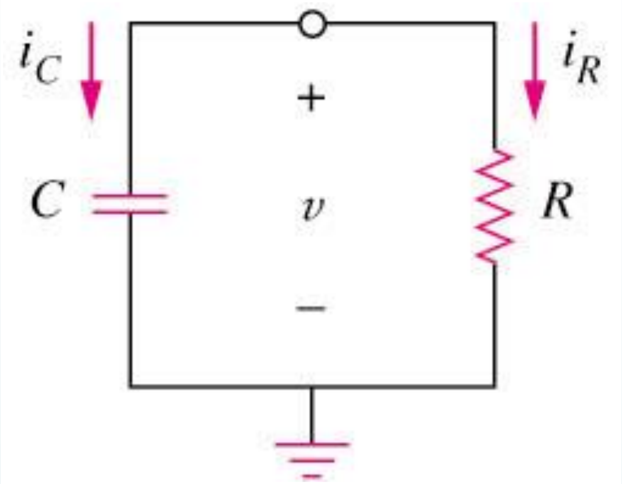


- The time constant τ of a circuit is the time required for the response to decay by a factor of $1/e$ or 36.8% of its initial value.
- v decays faster for small τ and slower for large τ.

7.1 The Source-Free RC Circuit (3)

The key to working with a source-free RC circuit is finding:

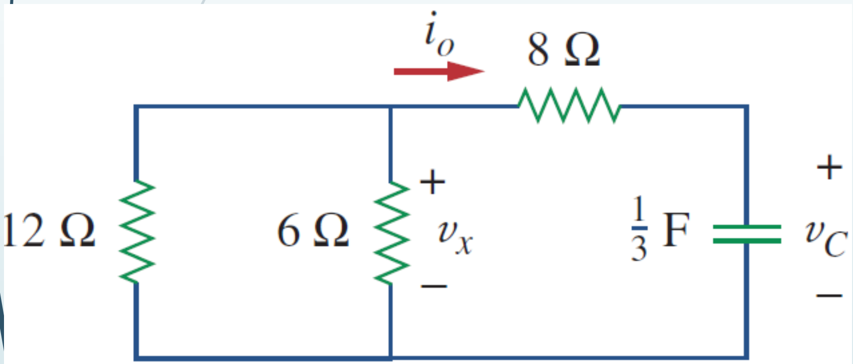
$v(t) = V_0 \, e^{-t/\tau}$ where $\tau = RC$

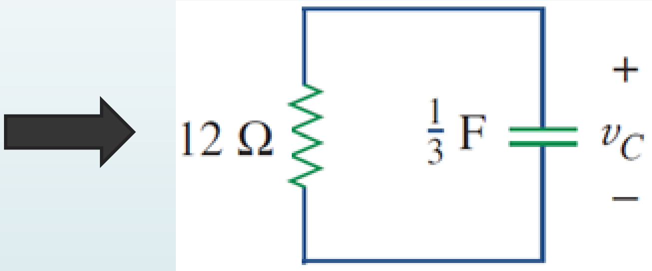


1. The initial voltage $v(0) = V_0$ across the capacitor.
2. $\tau = RC$

7.1 The Source-Free RC Circuit (4)

Ex.1 Refer to the circuit below, determine v_C , v_x , and i_o for $t \geq 0$. Assume that $v_C(0) = 60$ V.





$$v_C(t) = V_o e^{-\frac{t}{\tau}} = 60e^{-\frac{t}{4}}$$

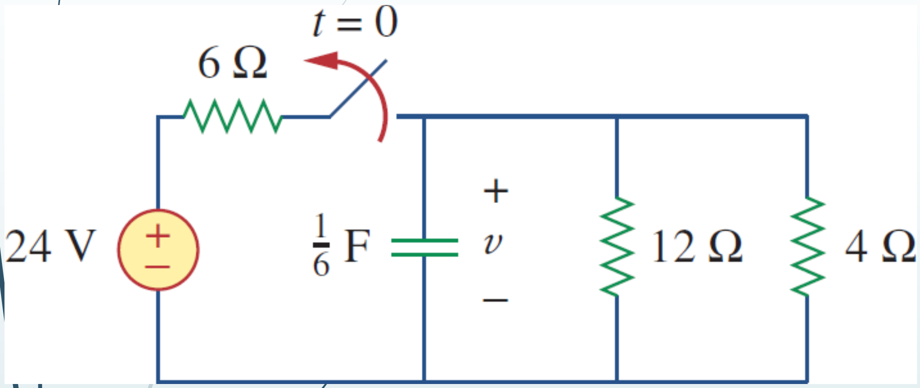
$$i_o(t) = C\frac{dv_C}{dt} = -5e^{-\frac{t}{4}}$$

$$v_x(t) = 8i_o + v_C(t)$$
$$= -40e^{-\frac{t}{4}} + 60e^{-\frac{t}{4}} = 20e^{-\frac{t}{4}}$$

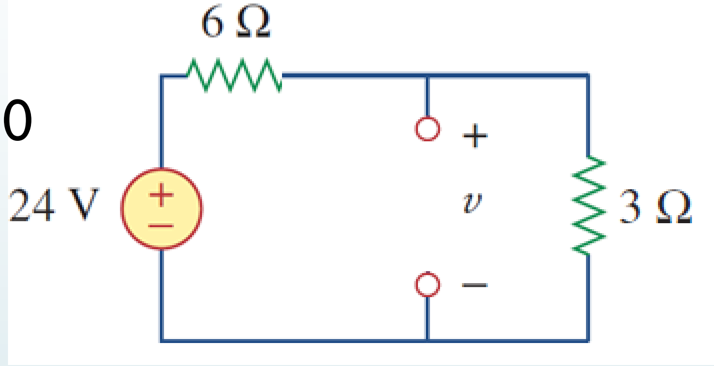
Answer: $v_C = 60e^{-0.25t}$ V ; $v_x = 20e^{-0.25t}$ V; $i_o = -5e^{-0.25t}$ A

7.1 The Source-Free RC Circuit (5)

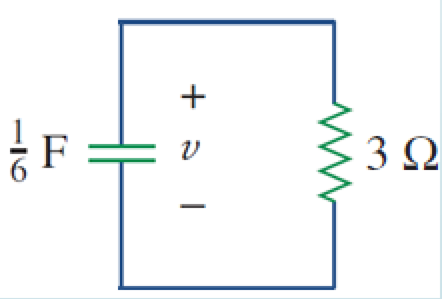
Ex.2 The switch is opened at $t = 0$, find $v(t)$ for $t \geq 0$.



SW @ *t*=0



SW @ *t* >0

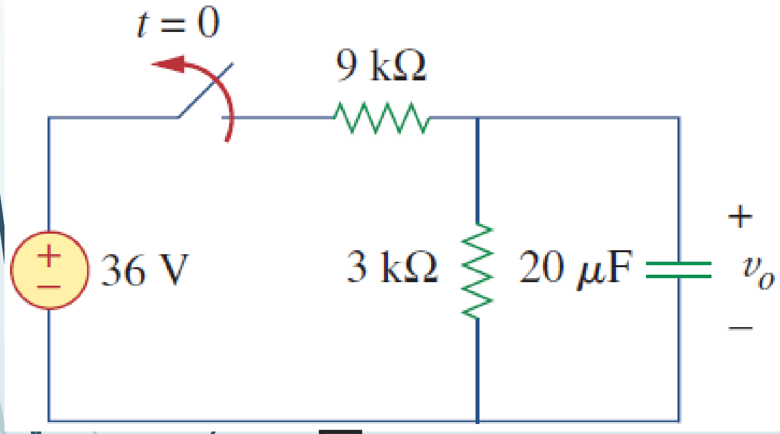


$$v(0) = V_0 = ? \rightarrow v(0) = \left(\frac{3}{3+6}\right)24 = 8\,V$$

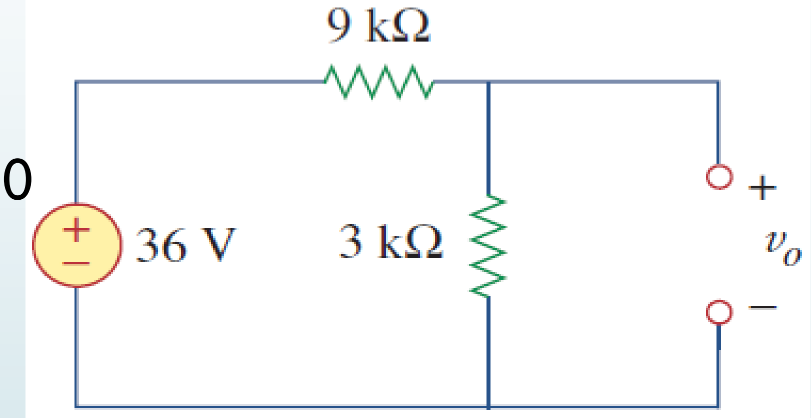
Answer: $v(t) = 8e^{-2t}$ V

7.1 The Source-Free RC Circuit (5)

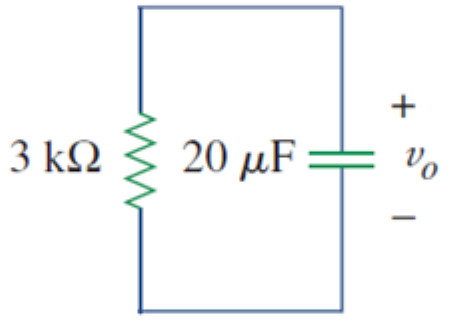
Ex.3 Find for $v_o(t)$ for t>0. Determine the time for the capacitor voltage to decay to one-third of its value at t=0



SW @ *t*=0



SW @ *t* >0



$$v(0) = V_0 = ? \to v(0) = \left(\frac{3}{3+9}\right)36 = 9\ V$$

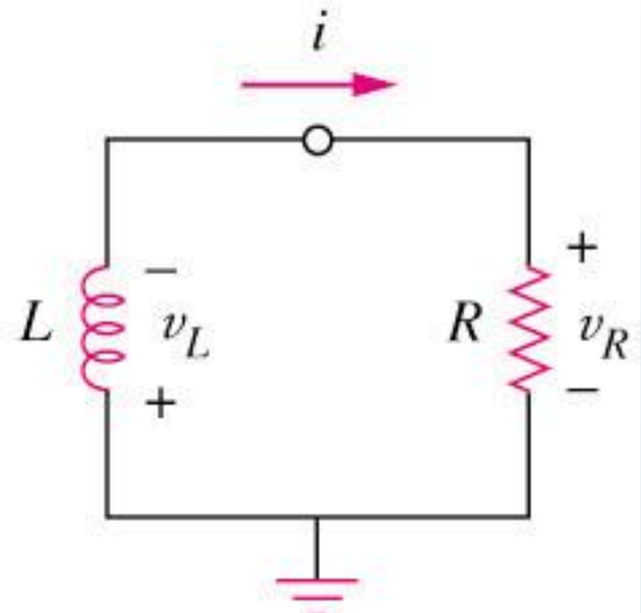
$$v(t) = 9e^{-t/0.06}\ \text{V}$$

$$9/3 = 3 = 9e^{-t/0.06}; t = -\,0.06 \times \ln(3/9) = 0.066\ s$$

Answer: $v(t) = 9e^{-t/0.06}$ V; t = 0.066 s

7.2 The Source-Free RL Circuit (1)

- A first-order RL circuit consists of L and R (or their equivalent)



By KVL

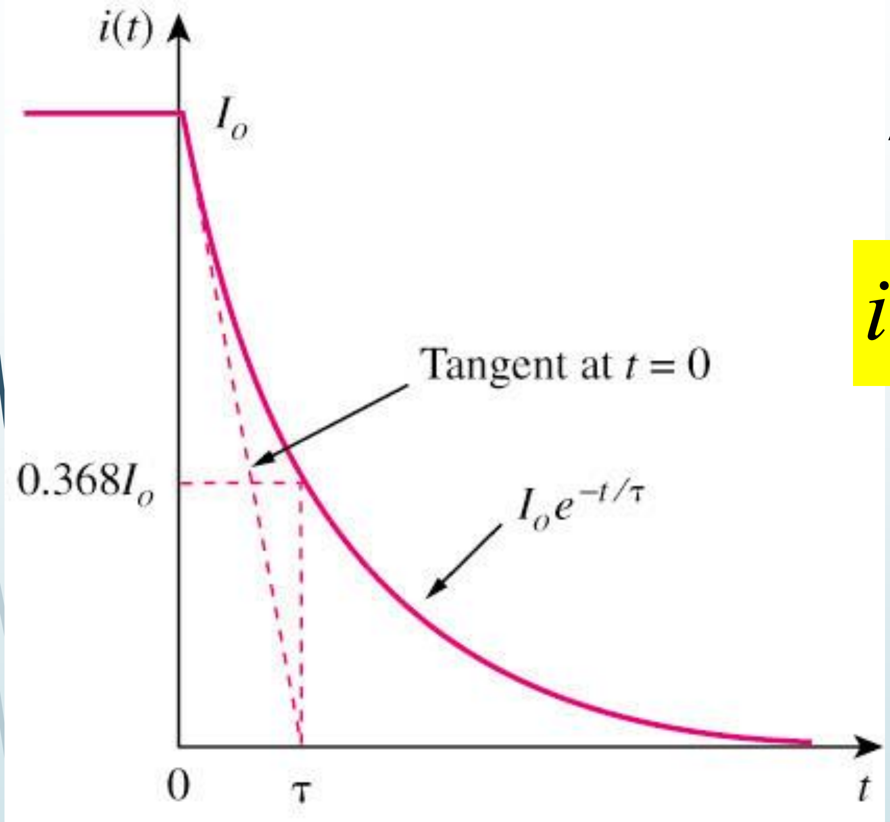
$$\boxed{v_L + v_R = 0} \rightarrow \boxed{L\frac{di}{dt} + iR = 0}$$

Inductors law

Ohms law

แก้สมการ 1storder differential eq. $\rightarrow \boxed{\frac{di}{i} = -\frac{R}{L}dt} \rightarrow \boxed{i(t) = I_0\, e^{-\left(\frac{R}{L}t\right)}}$

7.2 The Source-Free RL Circuit (2)



A general form representing RL

$$i(t) = I_0\, e^{-t/\tau} \quad \text{where} \quad \tau = \frac{L}{R}$$

- The time constant τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- $i(t)$ decays faster for small t and slower for large t.
- The general form is very similar to a RC source-free circuit.

7.2 The Source-Free RL Circuit (3)

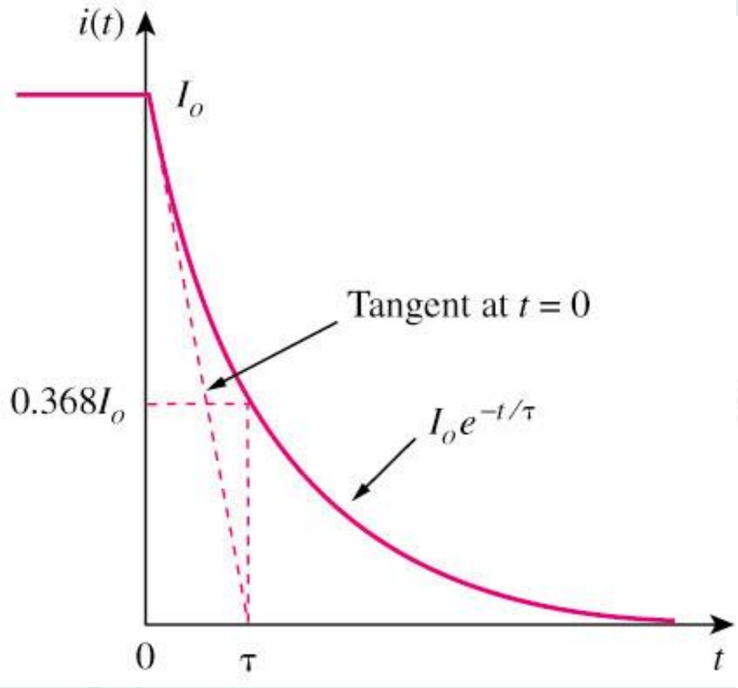
Comparison between RL and RC circuit

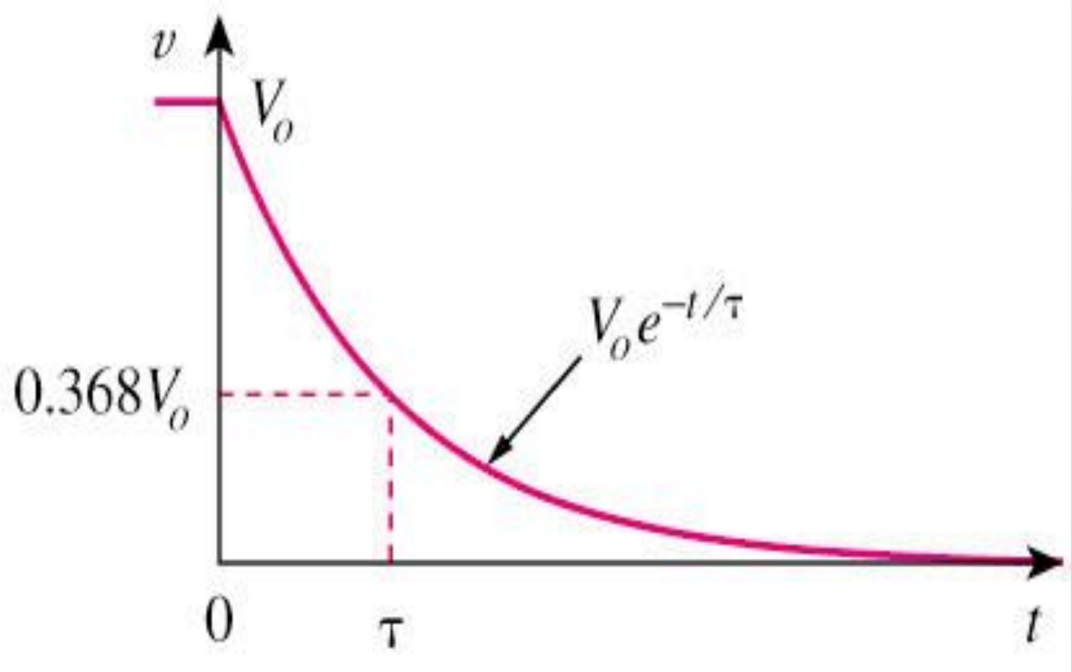
RL source-free circuit

$i(t) = I_0\, e^{-t/\tau}$ where $\tau = \dfrac{L}{R}$

RC source-free circuit

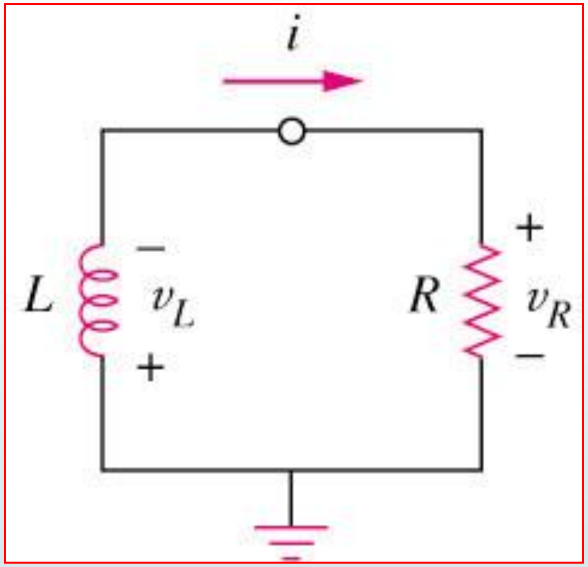
$v(t) = V_0\, e^{-t/\tau}$ where $\tau = RC$





7.2 The Source-Free RL Circuit (4)

The key to working with a source-free RL circuit is finding:

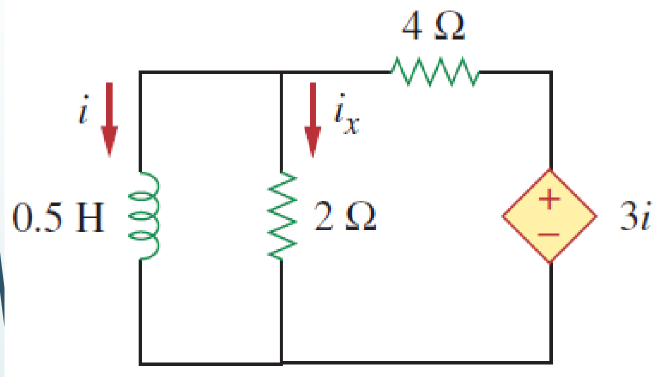


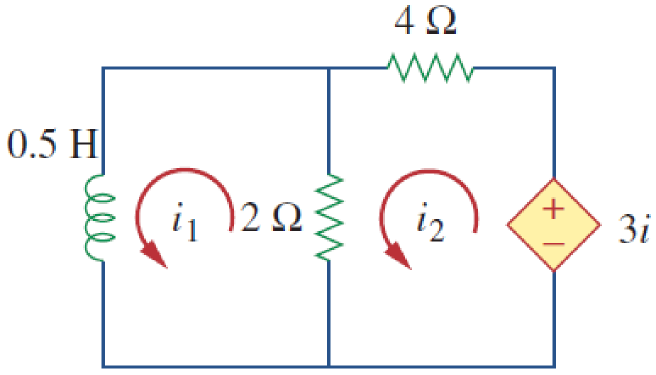
$$i(t) = I_0 \, e^{-t/\tau}$$ where $$\tau = \frac{L}{R}$$

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau = L/R$.

7.2 The Source-Free RL Circuit (5)

Ex.4 Assume that $i(0) = 10$ A. Find $i(t)$ and $i_x(t)$.





วิธีที่ 1 ใช้ KVL

$$\text{loop 1,} \quad \frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

$$\text{loop 2,} \quad 6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1$$

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0 \qquad \text{Since } i_1 = i$$

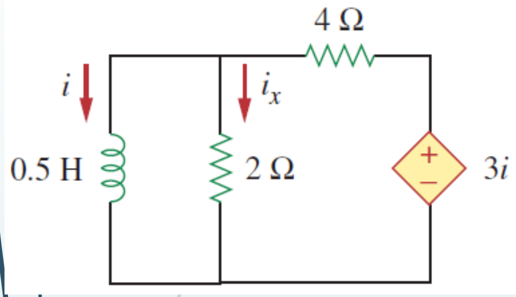
$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

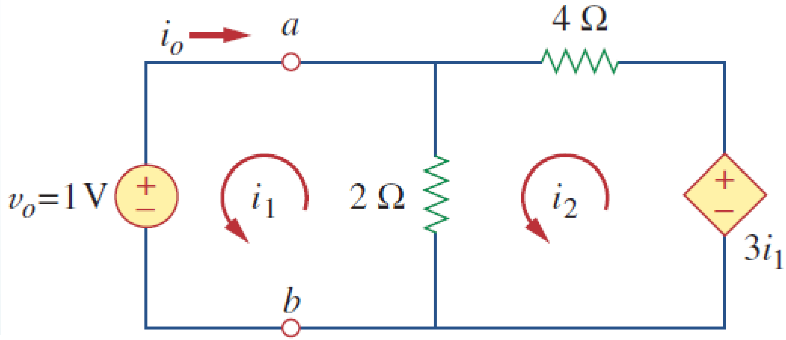
$$i_x(t) = i_2 - i_1 = -1.6667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Answer: $i_x(t) = 10e^{-2t/3}$ A; $i_x(t) = -1.6667e^{-2t/3}$ A

7.2 The Source-Free RL Circuit (5)

วิธีที่ 2 ใช้ R_{eq} และ KVL





$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1$$

$$i_1 = -3\text{ A}, \qquad i_o = -i_1 = 3\text{ A}$$

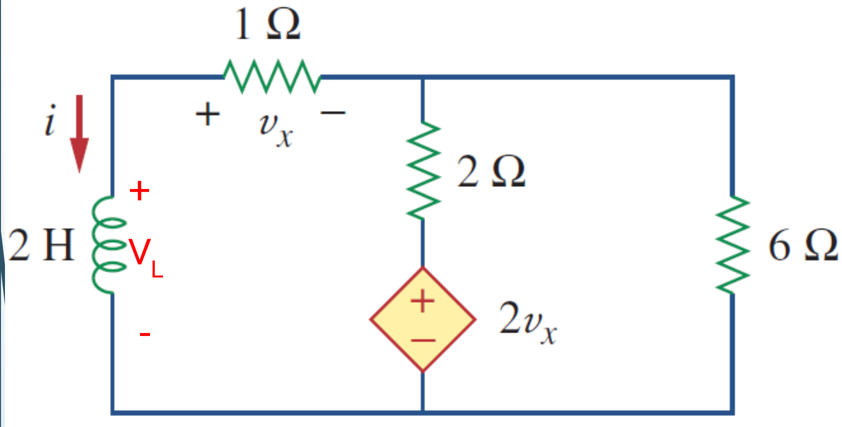
$$R_{\text{eq}} = R_{\text{Th}} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega \qquad \tau = \frac{L}{R_{\text{eq}}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

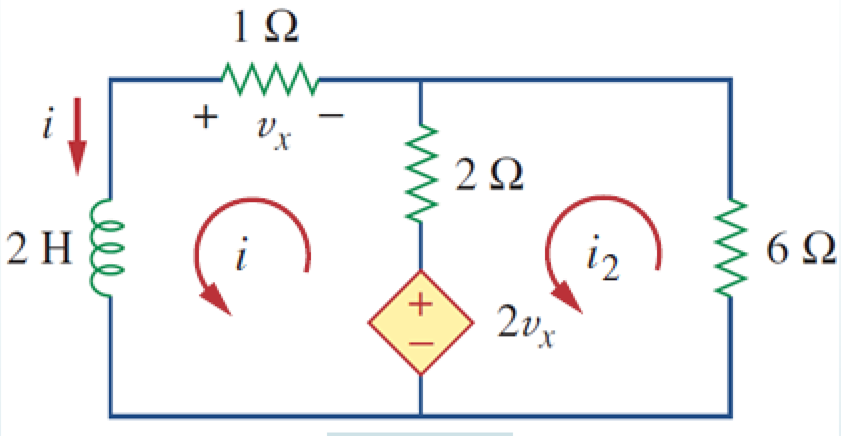
$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t}\text{ A}, \qquad t > 0$$

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t}\text{ A}, \qquad t > 0$$

7.2 The Source-Free RL Circuit (5)

Ex.5 Find i and v_x. Assume that $i(0) = 12$ A.





loop 1, $2\frac{di}{dt}+2(i-i_2)-3v_x=0$

$v_x=-i$

loop 2, $8i_2-2i+2v_x=0$, $8i_2+4v_x=0$, $i_2=-0.5\,v_x=0.5i$

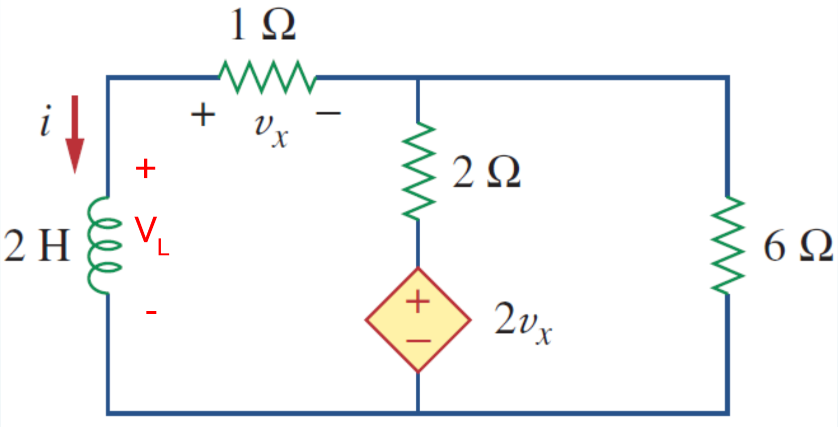
$$2\frac{di}{dt}+2(i-0.5i)+3i=0 \Rightarrow 2\frac{di}{dt}+4i=0 \Rightarrow \frac{di}{dt}+2i=0$$

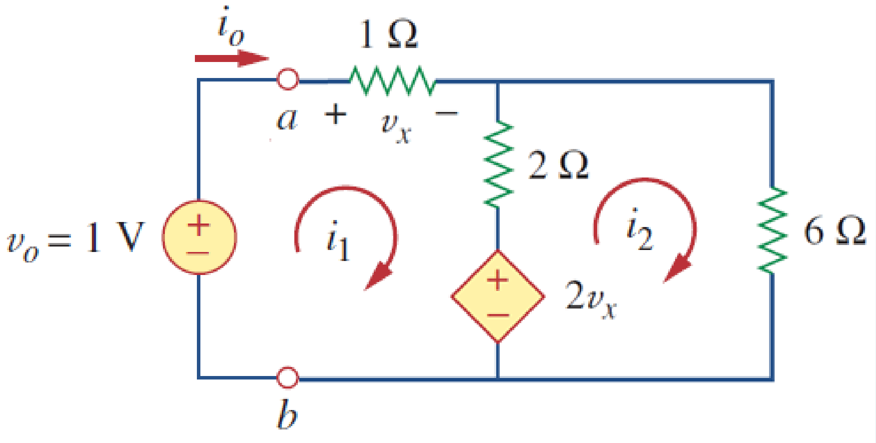
$$\frac{di}{i}=-2dt \Rightarrow \ln(i)=-2t \Rightarrow i=I_oe^{-2t}$$

Answer: $i(t) = 12e^{-2t}$ A; $v_x(t) = -i(t) = -12e^{-2t}$ V

7.2 The Source-Free RL Circuit (5)

Ex.5 Find i and v_x. Assume that $i(0) = 12$ A.





loop 1, $3i_1 - 2i_2 + 2i_1 = 5i_1 - 2i_2 = 1$

loop 2, $2i_1 = -2i_1 + 8i_2 \quad i_1 = 2i_2$

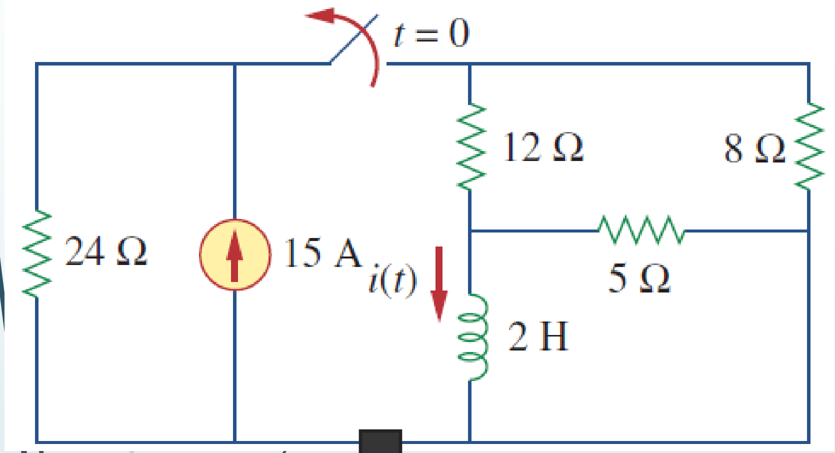
$$i_2 = \frac{1}{8}, i_1 = \frac{1}{4}$$

$$R_{eq} = \frac{v_o}{i_o} = 4\Omega, \quad \tau = \frac{L}{R_{eq}} = \frac{2}{4} = \frac{1}{2}$$

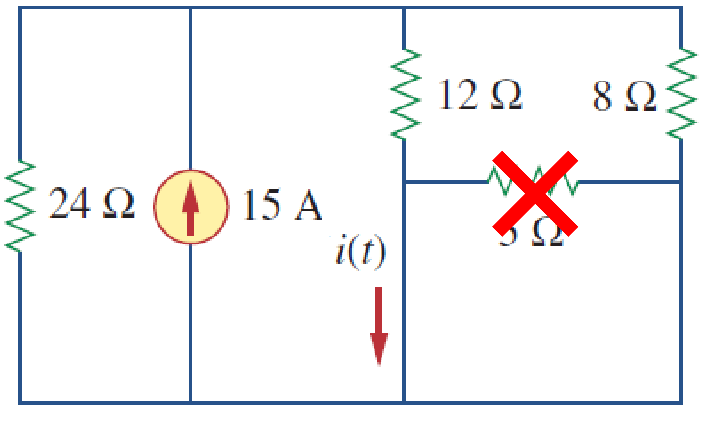
Answer: $i(t) = 12e^{-2t}$ A; $v_x(t) = -i(t) = -12e^{-2t}$ V

7.2 The Source-Free RL Circuit (6)

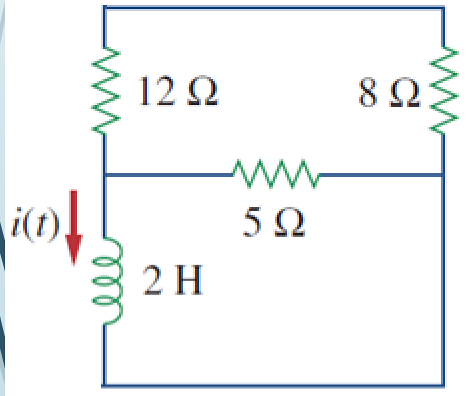
Ex.6 For the circuit, find ***i*** **(*t*)** for *t* > 0.



SW @ *t*=0



SW @ *t* >0



$$i(0) = I_0 = ? \rightarrow i(0) = \left(\frac{24//8}{(24//8)+12}\right)15 = 5\ \text{A}$$

$$R_{eq} = (12+8)//5 = 4$$

Answer: $i(t) = 5e^{-2t}$ A , $t > 0$

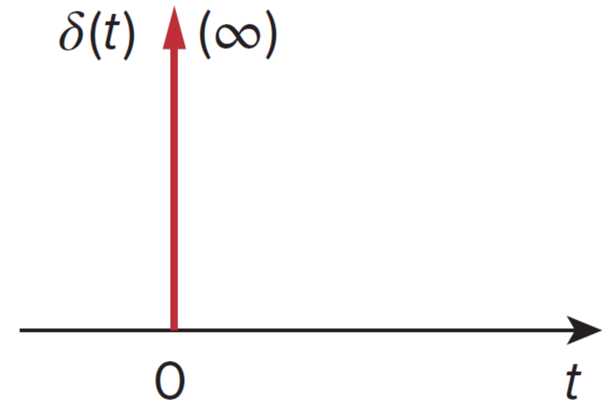
7.3 Singularity Functions (1)

- Singularity functions are functions that either are discontinuous or have discontinuous derivatives.
- 3 most widely used singularity functions in circuit analysis are the unit impulse $\delta(t)$, the unit ramp $r(t)$, and the unit step $u(t)$ functions.

7.3 Unit Impulse Function (2)

- The unit impulse function $\delta(t)$ is zero everywhere except at t = 0, where it is undefined.
- This may be expressed mathematically as

$$\int_{0^-}^{0^+} \delta(t)dt = 1$$

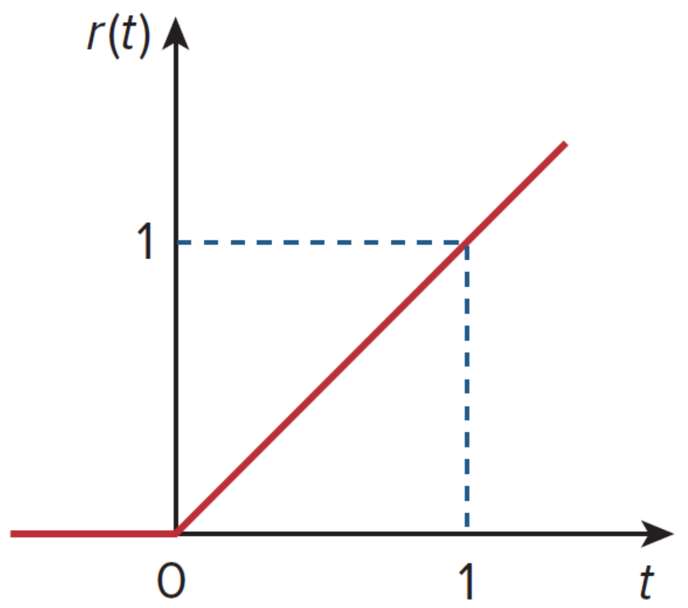


The unit impulse function.

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

7.3 Unit Ramp Function (3)

- The unit ramp function $r(t)$ is zero for negative values of t and has a unit slope for positive values of t.



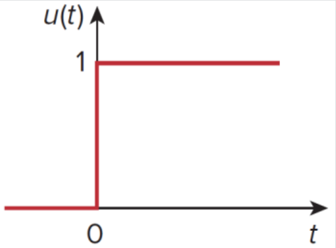
The unit ramp function.

$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

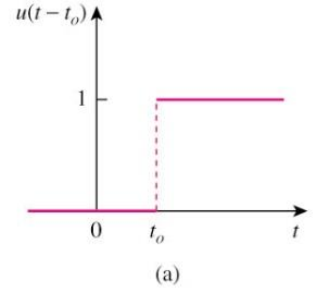
7.3 Unit Step Function (4)

- The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.

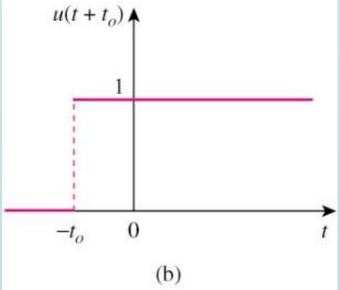
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t - t_o) = \begin{cases} 0, & t < t_o \\ 1, & t > t_o \end{cases}$$



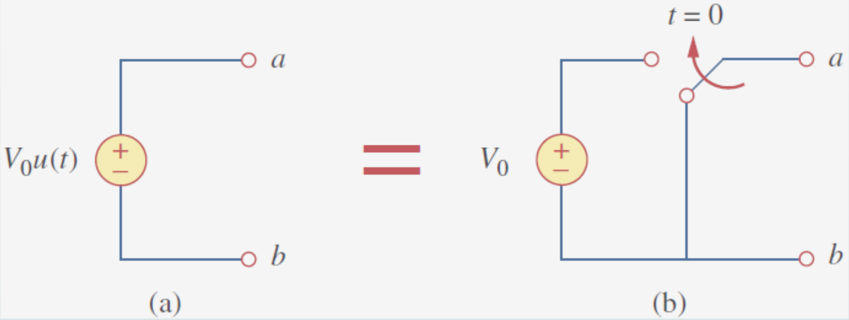
$$u(t + t_o) = \begin{cases} 0, & t < -t_o \\ 1, & t > -t_o \end{cases}$$



7.3 Unit Step Function (5)

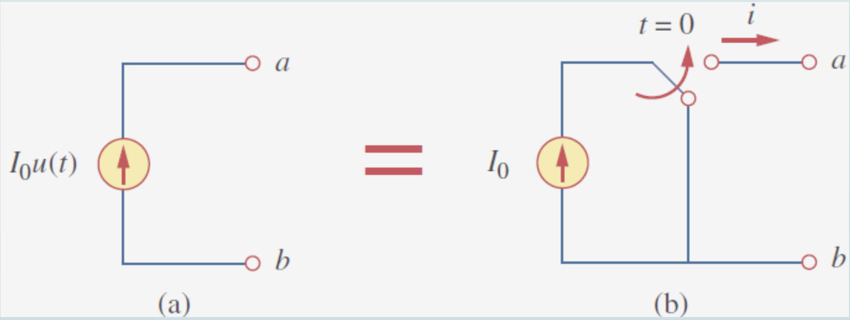
Represent an abrupt change for:

1. voltage source:



(a) (b)

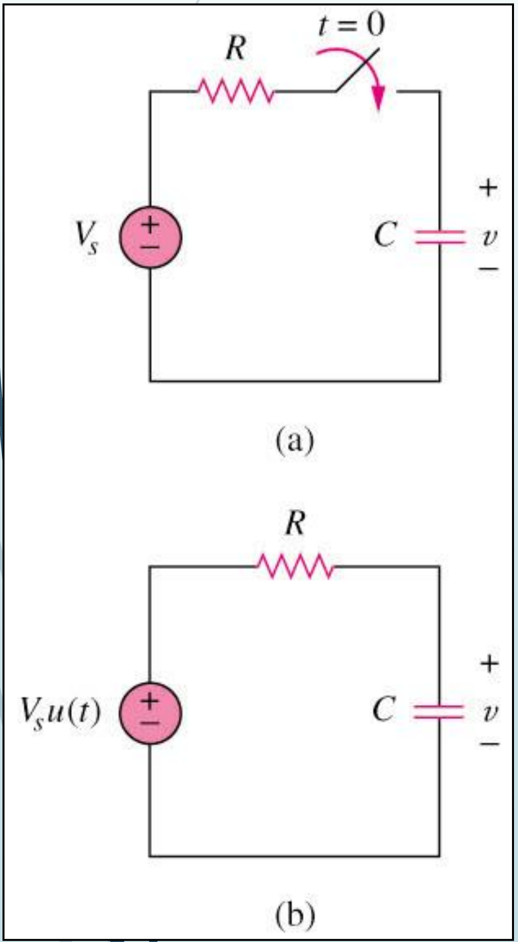
2. current source:



(a) (b)

7.4 The Step-Response of a RC Circuit (1)

- The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- Initial condition: $v(0^-) = v(0^+) = V_0$
- Applying KCL, $C\dfrac{dv}{dt} + \dfrac{v - V_s u(t)}{R} = 0$

or

$$\frac{dv}{dt} = -\frac{v - V_s u(t)}{RC}$$

- Where $u(t)$ is the unit-step function

$$\int \frac{dv}{v - V_s} = -\int \frac{dt}{RC}$$

$$\ln(v - V_s)\Big]_{V_0}^{v(t)} = -\frac{t}{RC}\Big]_0^t$$

7.4 The Step-Response of a RC Circuit (1)

$$\ln(v - V_s)\Big]_{V_0}^{v(t)} = -\frac{t}{RC}\Big]_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC}$$

$$\ln\left(\frac{v(t) - V_s}{V_0 - V_s}\right) = -\frac{t}{RC}$$

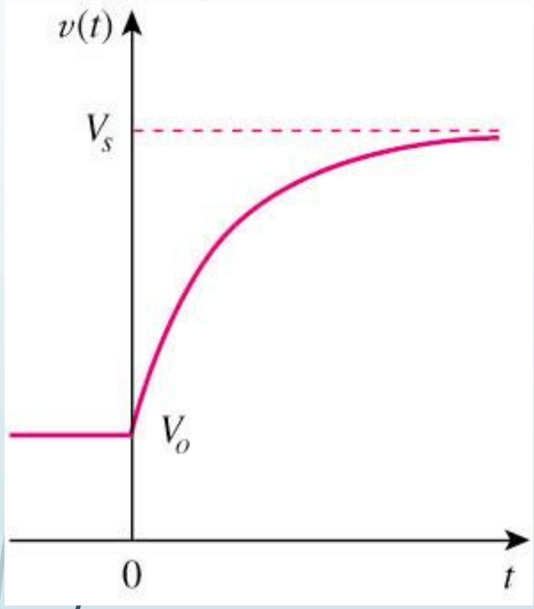
$$\frac{v(t) - V_s}{V_0 - V_s} = e^{-\frac{t}{RC}}$$

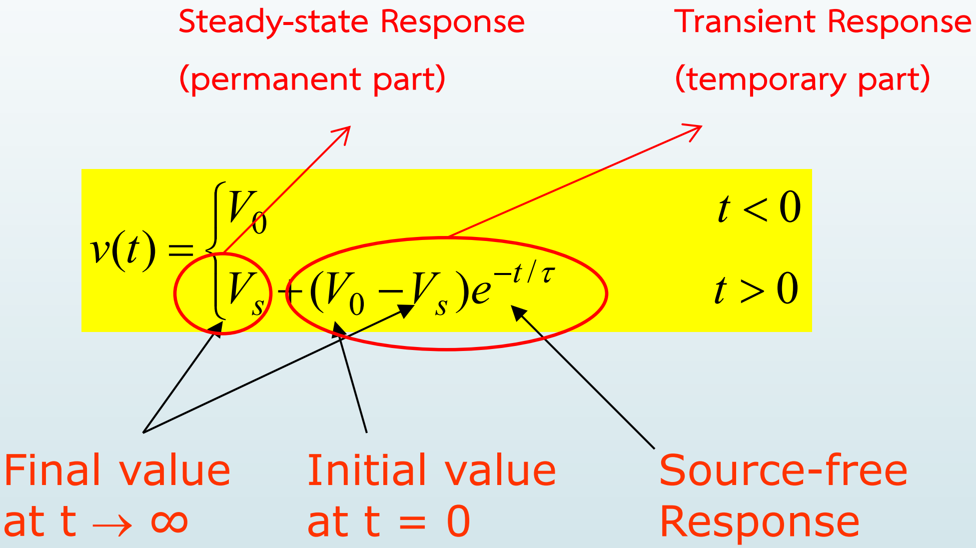
$$v(t) = V_s + \left(V_0 - V_s\right)e^{-\frac{t}{RC}}$$

$$v(t) = v(\infty) + \left(v(0) - v(\infty)\right)e^{-\frac{t}{\tau}}$$

7.4 The Step-Response of a RC Circuit (2)

- Integrating both sides and considering the initial conditions, the solution of the equation is:





Steady-state Response (permanent part)

Transient Response (temporary part)

$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}$$

Final value at t → ∞

Initial value at t = 0

Source-free Response

Complete Response = Natural response + Forced Response

(stored energy) (independent source)

(เกิดจาก no source) (เกิดจาก source)

$$V_0 e^{-t/\tau} \quad + \quad V_s(1 - e^{-t/\tau})$$

7.4 The Step-Response of a RC Circuit (3)

3 steps to find out the step response of an RC circuit:

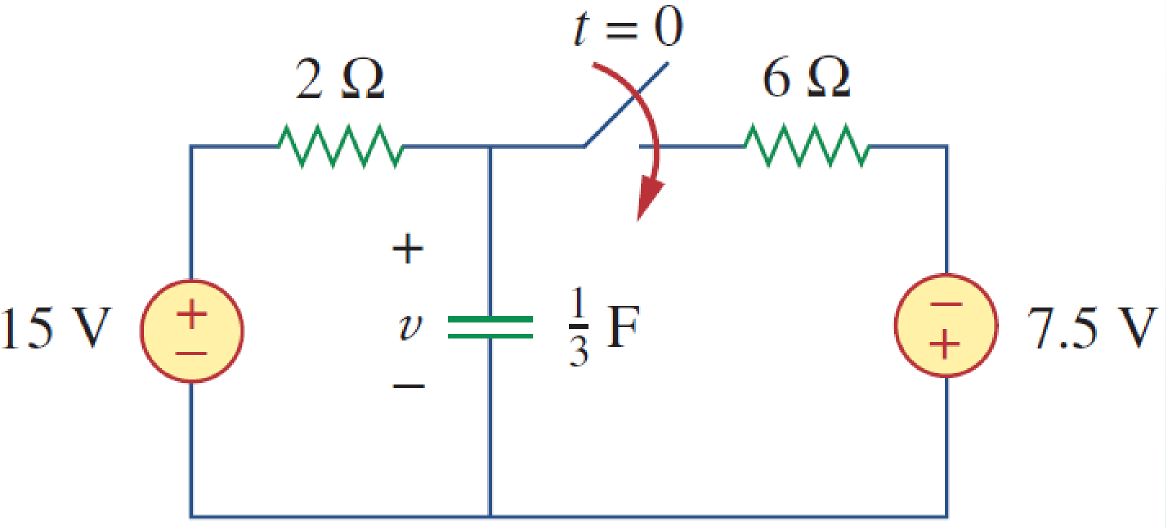
1. Initial capacitor voltage $v(0)$.
2. Final capacitor voltage $v(\infty)$ — DC voltage across C.
3. Time constant τ.

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

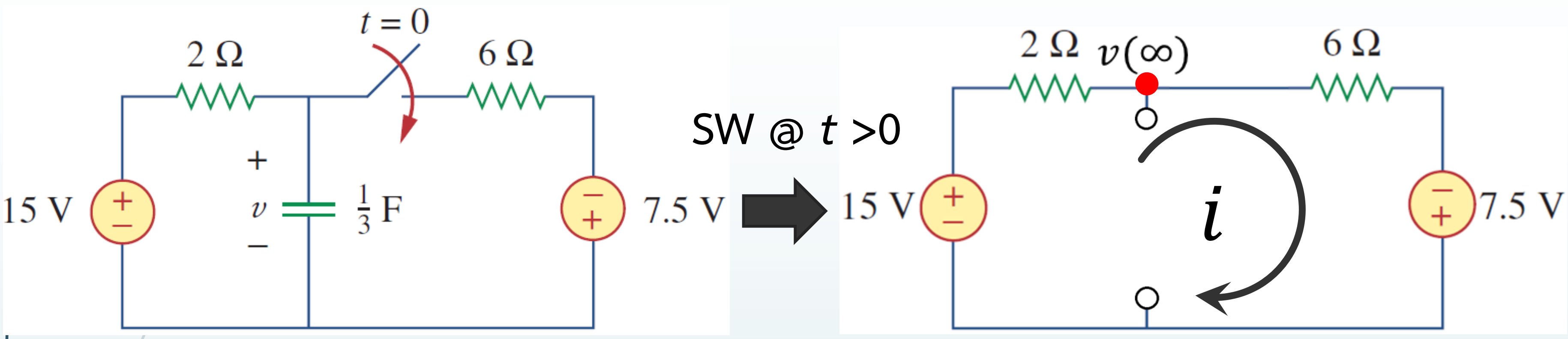
7.4 The Step-Response of a RC Circuit (4)

Ex.7 Find **v(t)** for t>0 in the circuit in below. Assume the switch has been open for a long time and is closed at t=0. Calculate **v(t)** at t=0.5.



Answer: $v(t) = (5.625e^{-2t} + 9.375)V$ เมื่อ t>0, $v\,(0.5) = 11.44\ V$

7.4 The Step-Response of a RC Circuit (5)



$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$v(0^-) = v(0^+) = 15 \text{ Volts}$$

$$i = \frac{15 + 7.5}{6 + 2} = 2.8125 \text{ A},$$

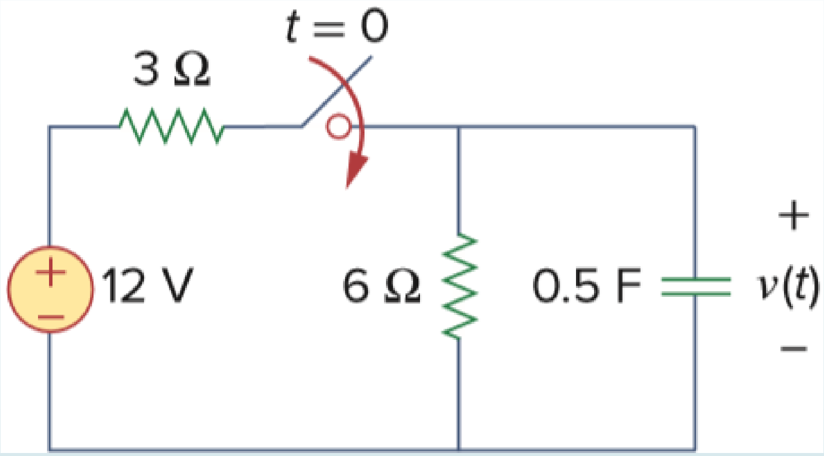
$$v(\infty) = 15 - (2.8125 \times 2) = 9.375\, Volts$$

$$\mathrm{R} = 2 // 6 = \frac{3}{2}\Omega, \qquad \mathrm{C} = \frac{1}{3}, \qquad RC = \frac{1}{2} \qquad -\frac{t}{\tau} = -\frac{t}{RC} = -2t$$

Answer: $v(t) = (5.625e^{-2t} + 9.375)V$ เมื่อ t>0, $v(0.5) = 11.44\ V$

7.4 The Step-Response of a RC Circuit (6)

Ex.8 Find $v(t)$ for t>0 in the circuit in below. Assume the switch has been open for a long time and is closed at t=0.



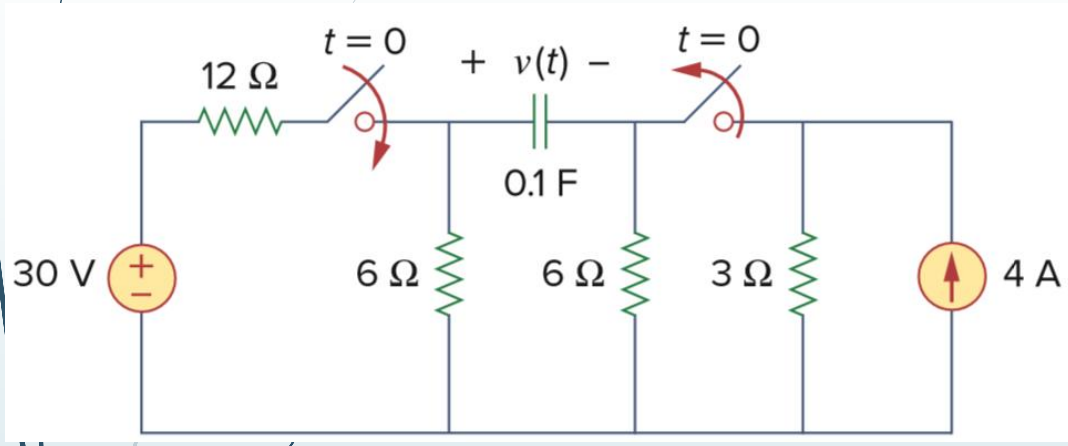
$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

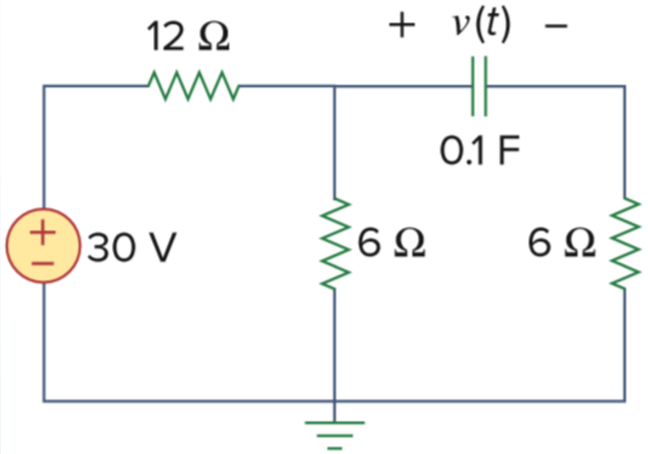
$$v\,(t) = \left(\frac{6}{3+6}\right)12 + [0 - \left(\frac{6}{3+6}\right)12]\,e^{-t/\tau}$$

Answer : $v(t) = 8(1 - e^{-t})$ V, $t > 0.$

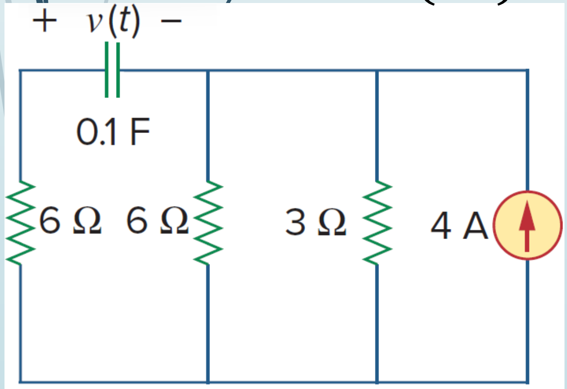
7.4 The Step-Response of a RC Circuit (7)

Ex.9 Find $v(t)$ for t>0 in the circuit in below.

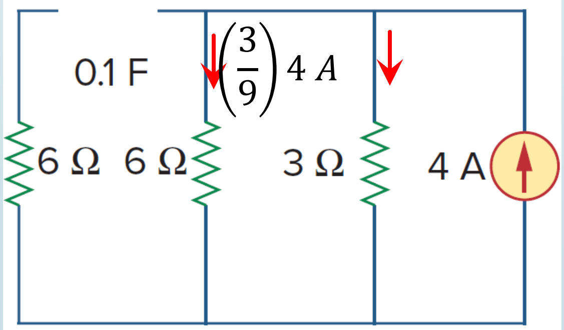




เริ่มต้นหา $v(o^-)$



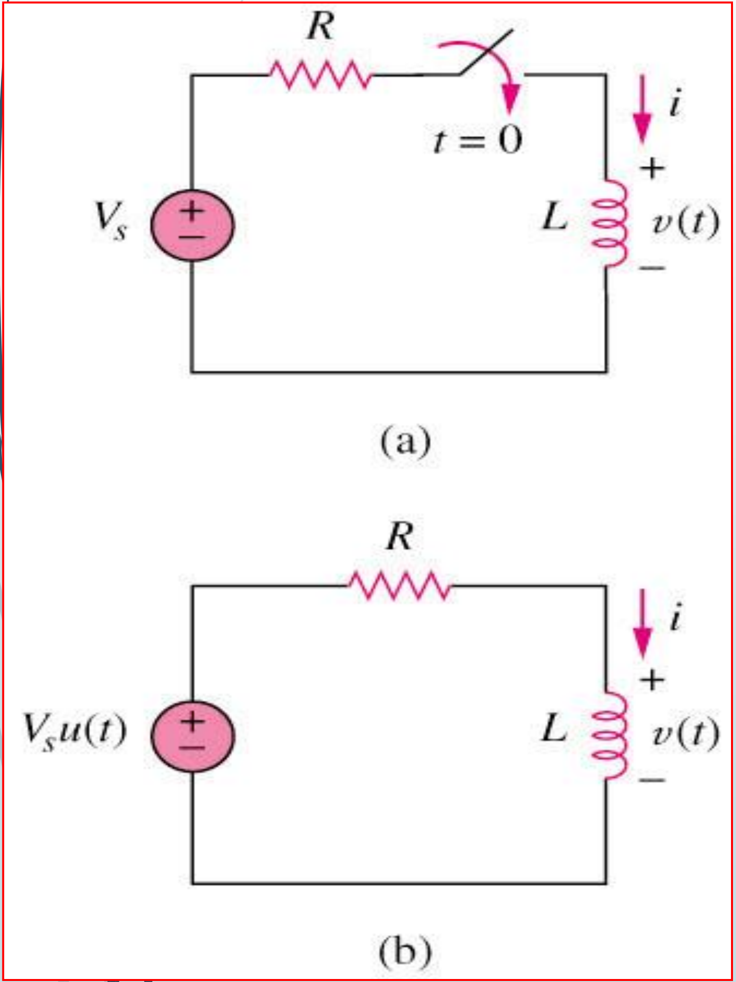
$$v(o^-) = -\left(\frac{3}{9}\right)4 \times 6 = -8\ V$$



Answer : $v(t) = 10 - 18e^{-t}$ V

7.5 The Step-response of a RL Circuit (1)

- The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- Initial current $i(0^-) = i(0^+) = I_0$
- Final inductor current

$$i(\infty) = V_s / R$$

- Time constant $\tau = L/R$

$$i(t) = \frac{V_s}{R} + (I_o - \frac{V_s}{R})e^{-\frac{t}{\tau}}$$

$$i(t) = i(\infty) + \left(i(0^+) - i(\infty)\right)e^{-\frac{t}{\tau}}$$

7.5 The Step-Response of a RL Circuit (2)

3 steps to find out the step response of an RL circuit:

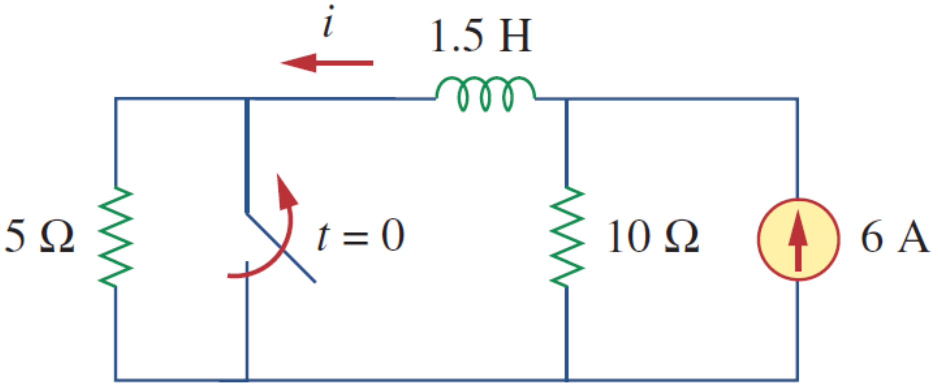
1. Initial inductor current $i(0)$ at $t = 0^+$.
2. Final inductor current $i(\infty)$.
3. Time constant τ.

$$i(t) = i(\infty) + [i(0+) - i(\infty)]\, e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

7.5 The Step-Response of a RL Circuit (3)

Ex.10 The switch in the circuit shown below has been closed for a long time. It opens at t = 0. Find i(t) for t > 0.



$$i(0) = \left(\frac{10}{10+0}\right)6 = 6\text{ A}$$

$$i(\infty) = \left(\frac{10}{5+10}\right)6 = 4\text{ A}$$

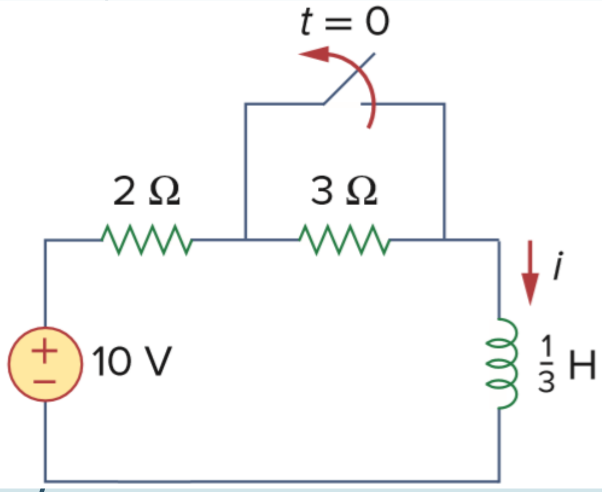
$$R_{eq} = 5+10 = 15\Omega \quad ;\tau = \left(\frac{L}{R_{eq}}\right) = \frac{1.5}{15} = \frac{1}{10}$$

$$i(t) = i(\infty) + [i(0+) - i(\infty)]\, e^{-t/\tau} = 4 + (6-4)e^{-10t} = 4 + 2e^{-10t}$$

Answer: $i(t) = 4 + 2e^{-10t}$

7.5 The Step-Response of a RL Circuit (4)

Ex.11 The switch in the circuit shown below has been closed for a long time. It opens at t = 0. Find i(t) for t > 0.



$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

$$i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

$$R_{\text{Th}} = 2 + 3 = 5\ \Omega$$

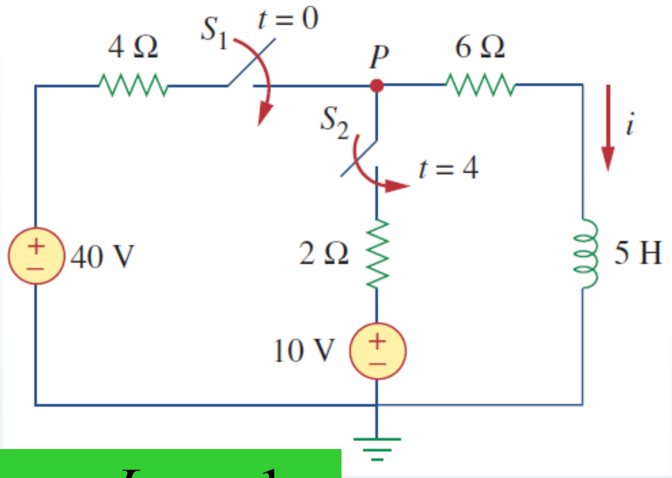
$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$= 2 + (5 - 2)e^{-15t} = 2 + 3e^{-15t} \text{ A},$$

Answer: $i(t) = 2 + 3e^{-15t}$ A

7.5 The Step-Response of a RL Circuit (5)



Ex.12 At t=0, S_1 is closed and S_2 is closed 4s later. Find i(t) for t=0. Calculate i for t=2s and t=5s.

แบ่งเป็น 3 ช่วง : $t < 0,\ 0 \le t \le 4,\ t > 4$

ช่วงที่ 1: $t < 0$: $i(0^-) = i(0^+) = 0$

ช่วงที่ 2: $0 \le t \le 4$: $i(4) = \dfrac{40}{4+6} = 4, R_{eq} = 10\Omega$ $\tau = \dfrac{L}{R_{eq}} = \dfrac{1}{2}s$

$$i(t) = i(4) + [i(0^+) - i(4)]e^{-t/\tau} = 4 + (0-4)e^{-2t} = 4(1 - e^{-2t})$$

ช่วงที่ 3: $t > 4$: $i(4) = i(4^-) = 4(1 - e^{-8}) \approx 4A$

$$KCL \text{ at node P}: \quad \frac{40-v}{4} + \frac{10-v}{2} = \frac{v}{6} \rightarrow v = \frac{180}{11}$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727A,\ R_{eq} = (4 // 2) + 6 = \frac{22}{3}\Omega \quad \tau = \frac{L}{R_{eq}} = \frac{15}{22}s$$

$$i(t)=i(\infty)+[i(4)-i(\infty)]e^{-(t-4)/\tau}=2.727+(4-2.727)e^{-(t-4)/\tau}=2.727+1.273e^{-1.4667(t-4)}$$

Answer: $i(2) = 4(1 - e^{-4}) = 3.93A, i(5) = 2.727 + 1.273e^{-1.4667} = 3.02A$