Electrical Engineering 1
12026105

Chapter 9

Sinusoids and Phasors

Learning Objectives

By using the information and exercises in this chapter you will be able to:

1. Better understand sinusoids.
2. Understand phasors.
3. Understand the phasor relationships for circuit elements.
4. Know and understand the concepts of impedance and admittance.
5. Understand Kirchhoff's laws in the frequency domain.
6. Comprehend the concept of phase-shift.
7. Understand the concept of AC bridges.

Sinusoids and Phasor Chapter 9

9.1 Motivation

9.2 Sinusoids' features

9.3 Phasors

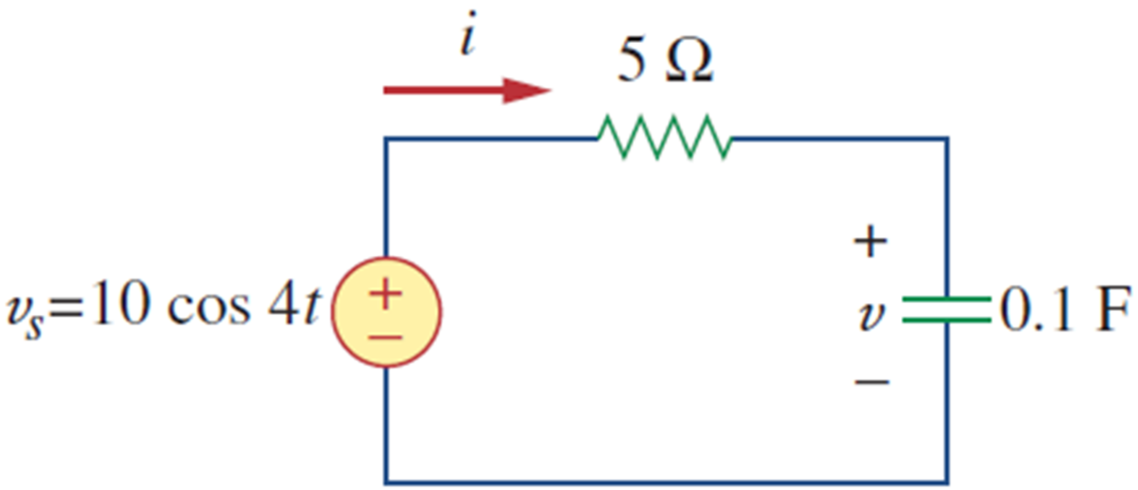
9.4 Phasor relationships for circuit elements

9.5 Impedance and admittance

9.6 Kirchhoff's laws in the frequency domain

9.7 Impedance combinations

9.1 Motivation



- How can we apply what we have learned before to determine $i(t)$ and $v(t)$?

9.2 Sinusoids (1)

A sinusoid is a signal that has the form of the *sine* or *cosine* function.

A general expression for the sinusoid, $v(t) = V_m \sin(\omega t + \phi)$, where

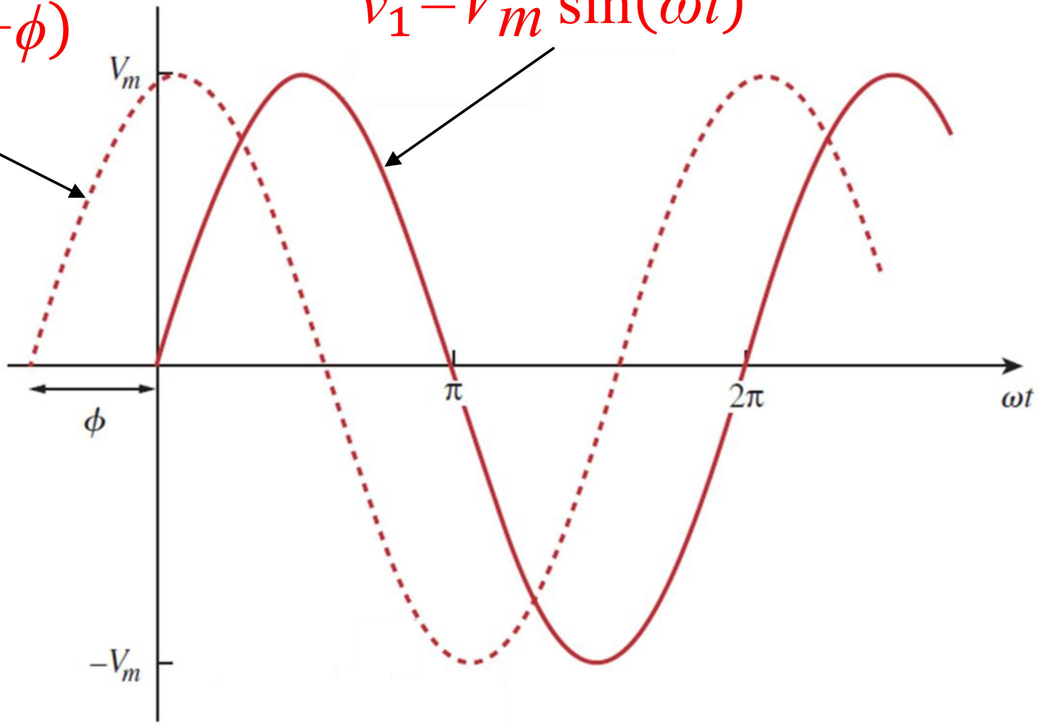
V_m = Amplitude of the sinusoid

ω = Angular frequency in radians/s $= 2\pi f$, $f = \frac{1}{T}$ [Hz]

ϕ = Phase

$v_2 = V_m \sin(\omega t + \phi)$

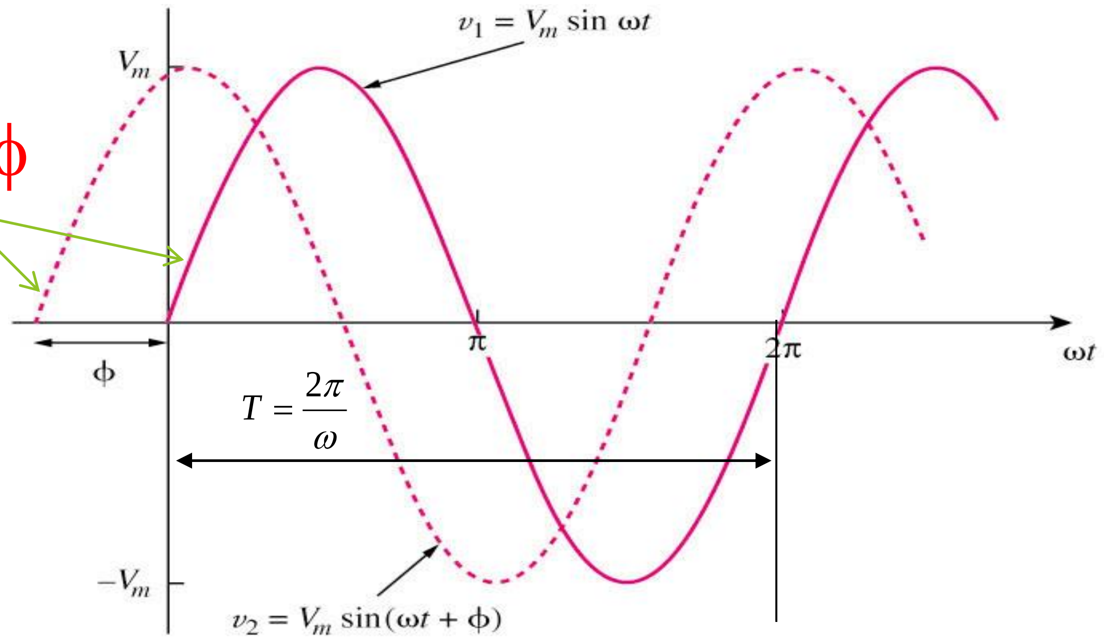
$v_1 = V_m \sin(\omega t)$



9.2 Sinusoids (2)

A periodic function is one that satisfies $v(t) = v(t + nT)$, for all t and for all integers n.

v_2 lead v_1 by ϕ



$$\omega = 2\pi f$$

$$f = \frac{1}{T} Hz$$

- Only 2 sinusoidal values with the <u>same freq.</u> can be compared.
- If phase difference is zero, they are in phase; otherwise, they are out of phase.

9.2 Sinusoids (3)

Example 1 Given a sinusoid, $5\sin(4\pi t - 60°)$, calculate its amplitude (V_m), phase (ϕ), angular frequency (ω), period (T), and frequency (f).

$$5\sin(4\pi t - 60°)$$

Amplitude=5

$$\omega = 4\pi = \frac{2\pi}{T}$$

Phase=-60°

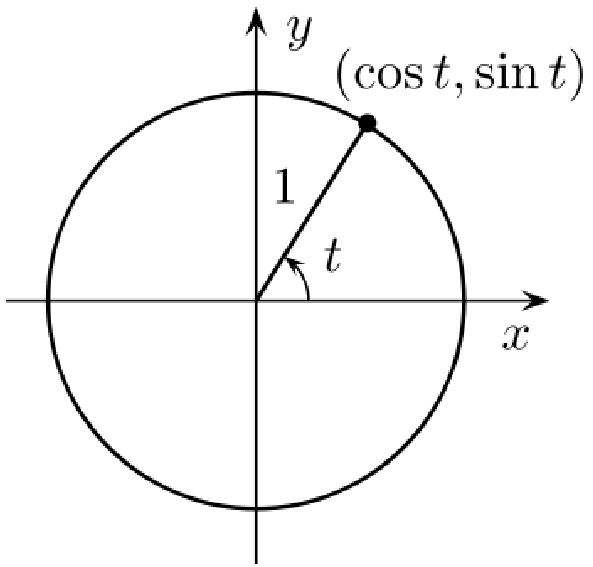
$\omega = 4\pi = 2\pi f$
$f = 2$ Hz
$T = 0.5$ s

Solution:

Amplitude = 5, phase = –60°, angular frequency = 4π rad/s, Period = 0.5 s, frequency = 2 Hz.

9.2 Sinusoids (4)

Example 2 Find the phase angle between $i_1 = -4\sin(377t+25°)$ and $i_2=5\cos(377t-40°)$, does i_1 lead or lag i_2?



Complementary Angle:

- $\sin\theta = \cos(90° - \theta)$
- $\cos\theta = \sin(90° - \theta)$
- $\tan\theta = \cot(90° - \theta)$
- $\cot\theta = \tan(90° - \theta)$
- $\sec\theta = \mathrm{cosec}(90° - \theta)$
- $\mathrm{cosec}\,\theta = \sec(90° - \theta)$

$$\sin(\theta+\pi) = -\sin\theta$$
$$\cos(\theta+\pi) = -\cos\theta$$
$$\sin(\theta+\tfrac{\pi}{2}) = +\cos\theta$$
$$\boxed{\cos(\theta+\tfrac{\pi}{2}) = -\sin\theta}$$

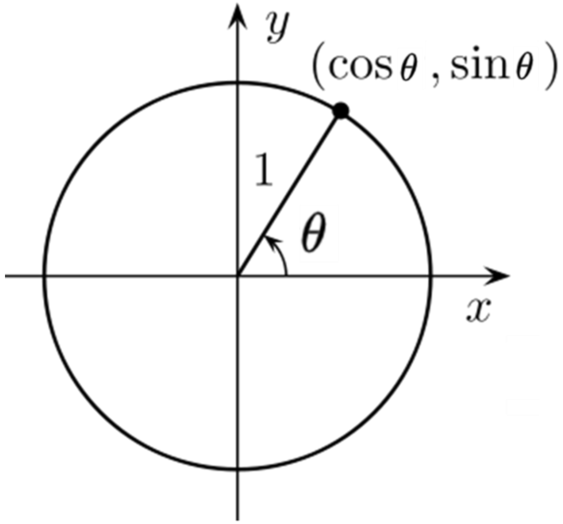
$i_1 = -4\sin(377t+25°) = 4\cos(377t+25° + 90°) = 4\cos(377t+115°)$

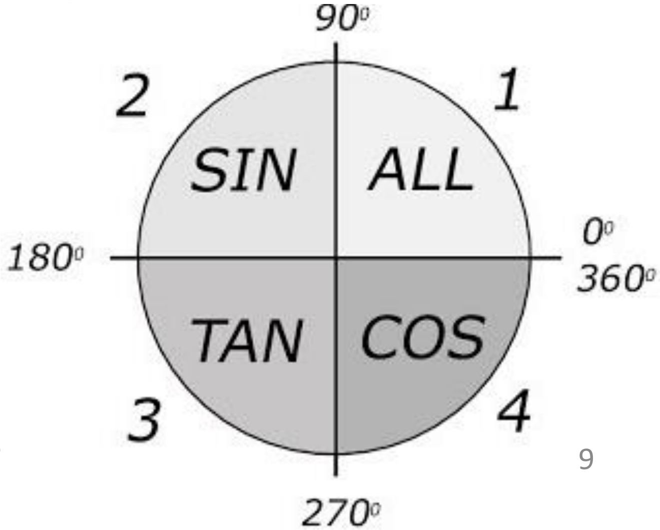
$i_2 = 5\cos(377t-40°)$ so that i_1 lead i_2 by $155°$

9.2 Sinusoids (5)

Reflected in $\theta = 0$	Reflected in $\theta = \pi/2$ (co-function identities)	Reflected in $\theta = \pi$
$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = +\cos\theta$	$\sin(\frac{\pi}{2} - \theta) = +\cos\theta$ $\cos(\frac{\pi}{2} - \theta) = +\sin\theta$	$\sin(\pi - \theta) = +\sin\theta$ $\cos(\pi - \theta) = -\cos\theta$

Shift by π/2	Shift by π Period for tan and cot	Shift by 2π Period for sin, cos
$\sin(\theta + \frac{\pi}{2}) = +\cos\theta$ $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$	$\sin(\theta + \pi) = -\sin\theta$ $\cos(\theta + \pi) = -\cos\theta$	$\sin(\theta + 2\pi) = +\sin\theta$ $\cos(\theta + 2\pi) = +\cos\theta$





- จงหาว่าในแต่ละคู่ของสัญญาณ sinusoids ตัวใดนำและนำกี่องศา

(a) $v(t) = 10\cos(4t - 60°)$ and
$i(t) = 4\sin(4t + 50°)$

(b) $v_1(t) = 4\cos(377t + 10°)$ and
$v_2(t) = -20\cos 377t$

Solution:

a. $i(t) = 4\,sin(4t - 40° + 90°) = 4\,cos(4t - 40°)$ ดังนั้น $i(t)$ นำ $v(t)$ อยู่ 20°

b. $v_2(t) = 20\,cos(377t + 180°)$ ดังนั้น $v_2(t)$ นำ $v_1(t)$ อยู่ 170°

- จงหาว่าในแต่ละคู่ของสัญญาณ sinusoids ตัวใดนำและนำกี่องศา

$$(c)\ x(t) = 13\cos 2t + 5\sin 2t \text{ and}$$
$$y(t) = 15\cos(2t - 11.8^\circ)$$

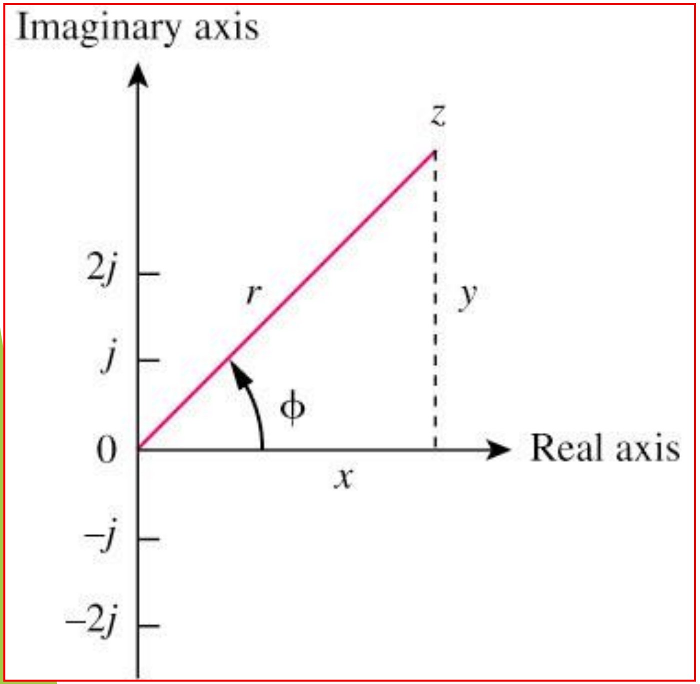
Solution:

$x(t) = 13\,cos(2t) + 5\,cos(2t - 90^\circ) = 13\angle 0^\circ + 5\angle - 90^\circ$

$= 13 - 5j = 13.93\angle - 21.04^\circ = 13.93\,cos(2t - 21.04^\circ)$ ดังนั้น $y(t)$นำ $x(t)$ อยู่ 9.24°

9.3 Phasor (1)

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- It can be represented in one of the following three forms:



a. Rectangular $z = x + jy = r(\cos\varphi + j\sin\varphi)$

b. Polar $z = r\angle\varphi$

c. Exponential $z = re^{j\varphi}$

where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\dfrac{y}{x}$

9.3 Phasor (2)

Example 3 Evaluate the following complex numbers:

$-13+j18$

$5(\cos(60°)+j\sin(60°)) = 2.5+j4.33$

a. $[(5+j2)(-1+j4)-5\angle 60^{o}]$

b. $\dfrac{10+j5+3\angle 40^{o}}{-3+j4}+10\angle 30^{o}$

$-0.3672-j2.7996$

$10(\cos(30°)+j\sin(30°)) = 8.66+j5$

Solution:

a. –15.5 + j13.67

b. 8.293 + j2.2

9.3 Phasor (3)

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\dfrac{1}{z} = \dfrac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugate $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

9.3 Phasor (4)

Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \longleftrightarrow \mathbf{V} = V_m \angle \phi$$

(time domain) (phasor domain)

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the cosine function in all our proceeding study.

9.3 Phasor (5)

Example 4 Transform the following sinusoids to phasors:

$$i(t) = 5\cos(50t - 40^\circ)\ \text{A}$$
$$v(t) = -4\sin(30t + 50^\circ)\ \text{V}$$

Answer : $I = 5\angle -40^\circ\ \ \text{A}$

เนื่องจาก $-\sin(\theta) = \cos(\theta + 90^\circ)$ ดังนั้น

$$v(t) = 4\cos(30t + 50^\circ + 90^\circ)\ \Rightarrow V = 4\angle 140^\circ\ \ \text{V}$$

9.3 Phasor (6)

Example 5 Transform the sinusoids corresponding to phasors:

a. $V = -10\angle 30^\circ$ V

b. $I = j(5 - j12)$ A

Solution:

a). $v(t) = 10\cos(\omega t + 210^\circ)$ V

Since $I = 12 + j5 = \sqrt{12^2 + 5^2}\angle\tan^{-1}(\frac{5}{12}) = 13\angle 22.62^\circ$

b). $i(t) = 13\cos(\omega t + 22.62^\circ)$ A

9.3 Phasor (7)

The differences between v(t) and **V**

- $v(t)$ is instantaneous or time-domain representation
  V is the frequency or phasor-domain representation.
- $v(t)$ is time dependent, **V** is not.
- $v(t)$ is always real with no complex term, **V** is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

9.3 Phasor (8)

Relationship between differential, integral operation in phasor listed as follow:

Time domain		Phasor domain
$v(t) = V_m \cos(\omega t + \varphi)$	$\longleftrightarrow$	$\mathbf{V} = V_m \angle \varphi$
$\dfrac{dv(t)}{dt}$	$\longleftrightarrow$	$j\omega\mathbf{V}$
$\displaystyle\int v(t)dt$	$\longleftrightarrow$	$\dfrac{\mathbf{V}}{j\omega}$

$$v(t) = V_m \cos(\omega t + \varphi) = Re\left(V_m e^{j\varphi} e^{j\omega t}\right) = Re\left(\mathbf{V} e^{j\omega t}\right) \leftrightarrow \quad \mathbf{V} = V_m \angle \varphi = V_m e^{j\varphi}$$

9.3 Phasor (9)

Example 6 Use phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation.

$$4i+8\int i\mathrm{d}t-3\frac{\mathrm{d}i}{\mathrm{d}t}=50\cos(2t+75°)$$

$$\omega=2$$

$$4I+8\frac{I}{j\omega}-3j\omega I=50\angle 75°$$

$$4I-j4I-j6I=50\angle 75°$$

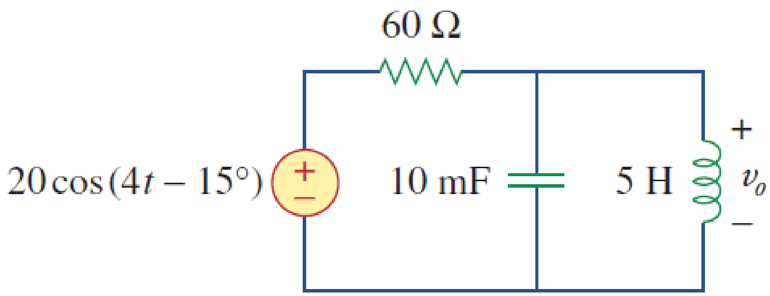
$$I(4-10j)=(10.77\angle-68.2°)I=50\angle 75°$$

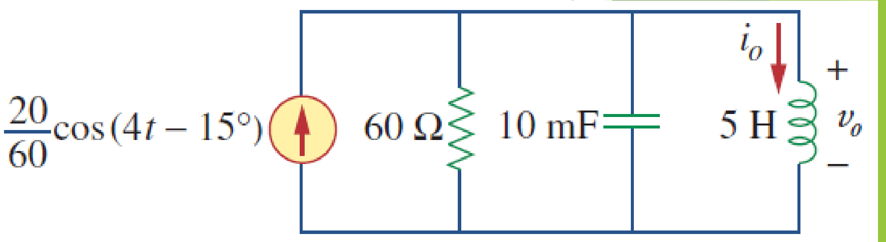
$$I=4.642\angle 143.2° \quad \Leftrightarrow \quad i(t)=4.642\cos(2t+143.2°)$$

Answer: $i(t)=4.642\cos(2t+143.2°)$ A

9.3 Phasor (10)

We can derive the differential equations for the following circuit to solve for $v_o(t)$ in phasor domain V_o.





$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \longrightarrow \frac{d^2 i_o}{dt^2} + \frac{5}{3}\frac{di_0}{dt} + 20 i_0 = \frac{20}{3}\cos(4t - 15^o)$$

$$v_o = L\frac{di_o}{dt}$$

- However, the derivation may sometimes be very tedious.

Is there any quicker and more systematic methods to do it?

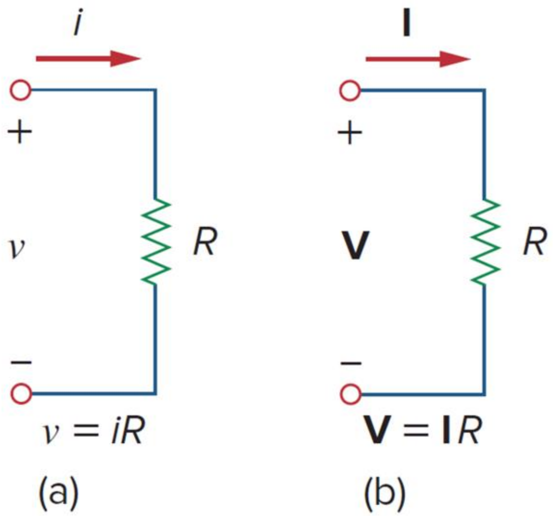
9.3 Phasor (11)

The answer is YES!

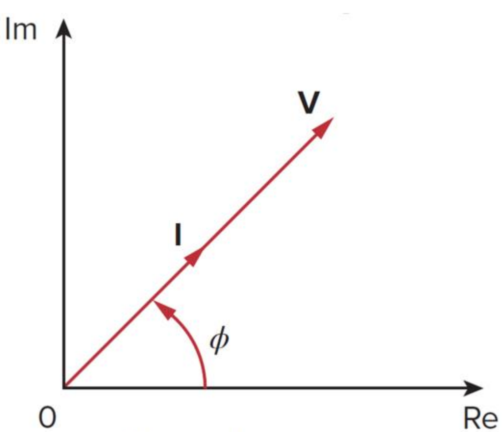
- Instead of first deriving the differential equation and then transforming it into phasor to solve for V_o,
- we can transform all the RLC components into phasor first, then apply the KCL laws and other theorems to set up a phasor equation involving V_o directly.

9.4 Phasor Relationships for Circuit Elements (1)

Resistor:

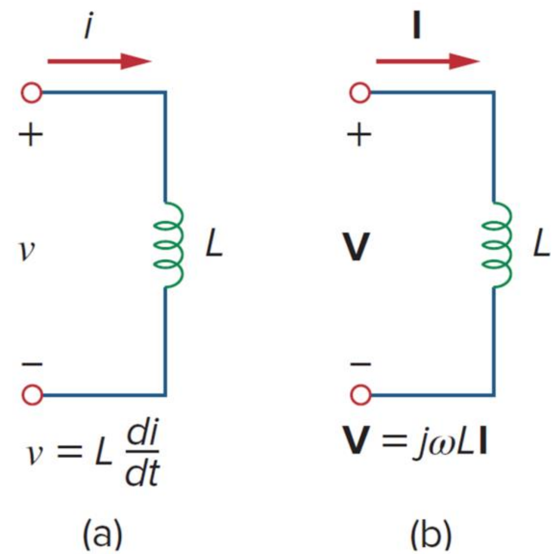


Voltage-current relations

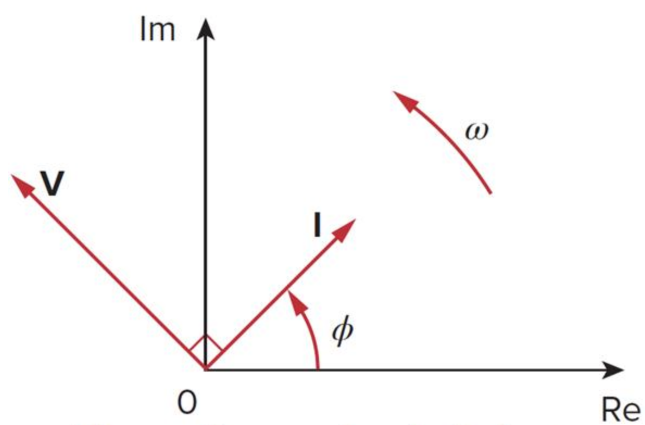


Phasor diagram

Inductor:

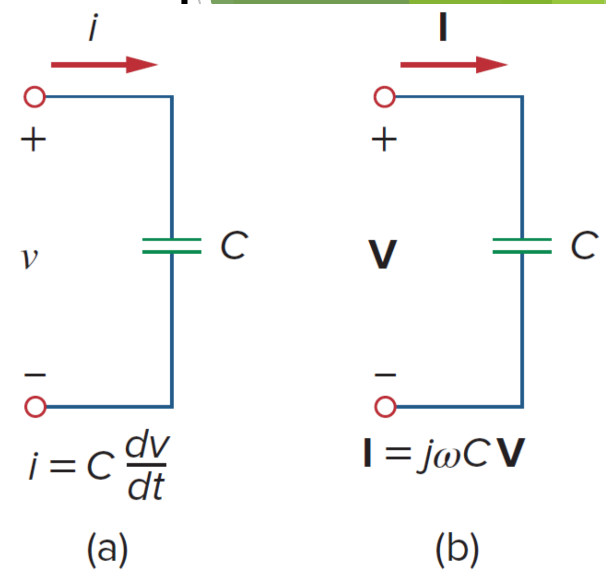


Voltage-current relations

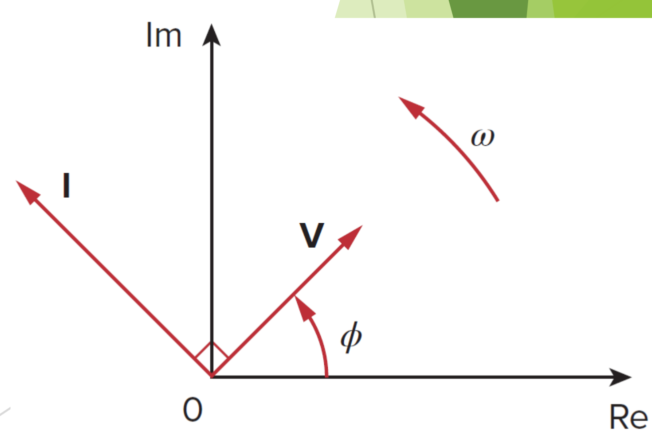


Phasor diagram for the inductor; **I** lags **V**.

Capacitor:



Voltage-current relations



Phasor diagram for the capacitor; **I** leads **V**.

9.4 Phasor Relationships for Circuit Elements (2)

Voltage-Current relationship		
Element	Time domain	Frequency domain
R	$v = iR$	$V = IR$
L	$v = L\dfrac{di}{dt}$	$V = j\omega LI$
C	$i = C\dfrac{dv}{dt}$	$I = j\omega CV$

9.4 Phasor Relationships for Circuit Elements (3)

Example 7 If $v(t) = 6\cos(100t - 30^{o})$ is applied to a 50 μF capacitor, calculate the current, $i(t)$, through the capacitor.

$$\mathrm{I} = j\omega \mathrm{CV}$$

$$\mathrm{I} = j \times 100 \times 50 \times 10^{-6} \times 6\angle -30^\circ = 0.03\angle 90^\circ \times \angle -30^\circ = 0.03\angle 60^\circ$$

$$i(t) = 0.03\cos(100t + 60^\circ)\ \mathrm{A}$$

Answer: $i(t) = 30\cos(100t + 60^{o})$ mA

9.5 Impedance and Admittance (1)

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in Ω.

$$Z = \frac{V}{I} = R + jX$$

where $R = Re(Z)$ is the resistance and $X = Im(Z)$ is the reactance. Positive X is for L and negative X is for C.

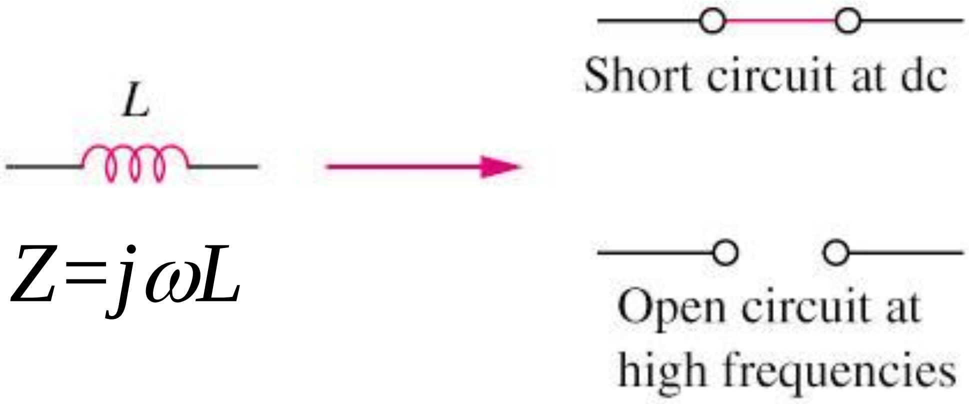
- The admittance Y is the reciprocal of impedance (Z),

measured in siemens (S). $Y = \frac{1}{Z}$

9.5 Impedance and Admittance (2)

Impedances of elements		
Element	**Impedance**	**Admittance**
R	$Z=R$	$Y=\dfrac{1}{R}$
L	$Z=j\omega L$	$Y=\dfrac{1}{j\omega L}$
C	$Z=\dfrac{1}{j\omega C}$	$Y=j\omega C$

9.5 Impedance and Admittance (3)

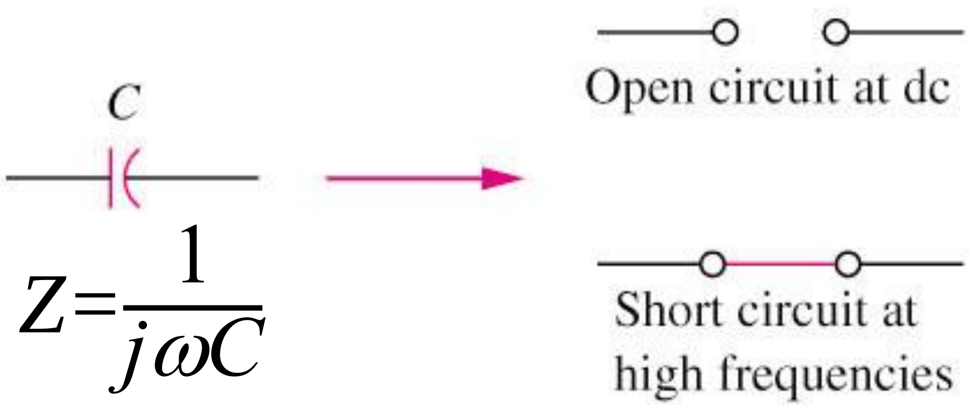


$$Z = j\omega L$$

$$\omega = 0; Z = 0$$

$$\omega \to \infty; Z \to \infty$$

(a)



$$Z = \frac{1}{j\omega C}$$

$$\omega = 0; Z \to \infty$$

$$\omega \to \infty; Z = 0$$

(b)

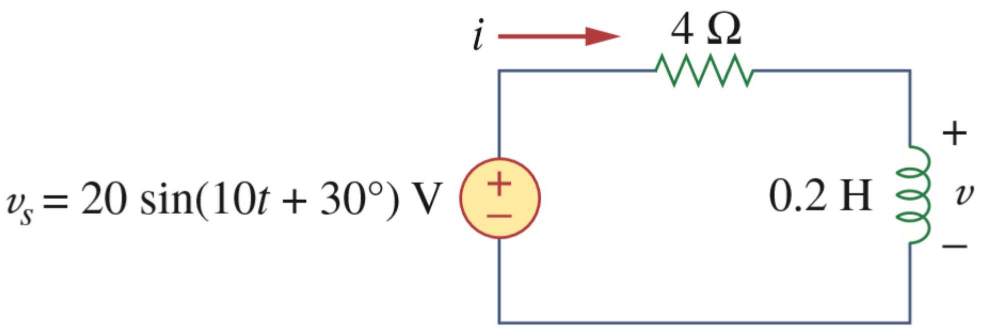
9.5 Impedance and Admittance (4)

After we know how to convert RLC components from time to phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.

Hence, we can apply the KCL laws and other theorems to directly set up phasor equations involving our target variable(s) for solving.

9.5 Impedance and Admittance (5)

Example 8 Refer to Figure below, determine $i(t)$ and $v(t)$.



$$v_s = 20\sin(10t+30°) = 20\cos(10t+30°-90°) \quad \Rightarrow \quad V_S = 20\angle -60°$$

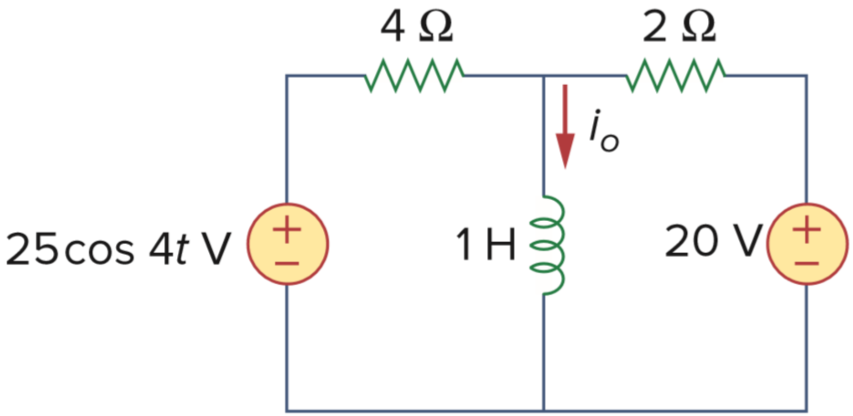
$$I = \frac{V_S}{Z} = \frac{20\angle -60°}{4+j2} \times \frac{4-j2}{4-j2} = -2\sqrt{5}\angle -86.57° \Rightarrow i = 4.472\cos(10t-86.57°) = 4.472\sin(10t+3.43°)$$

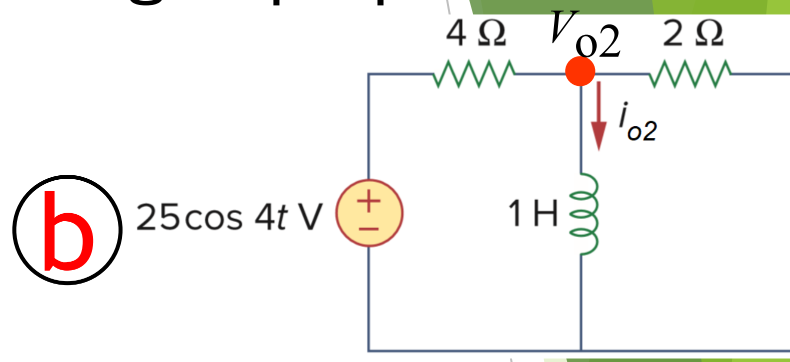
$$V = IZ_L = -2\sqrt{5}\angle -86.57° \times j2 = -4\sqrt{5}\angle 3.43° \Rightarrow v = 8.944\cos(10t+3.43°) = 8.944\sin(10t+93.43°)$$

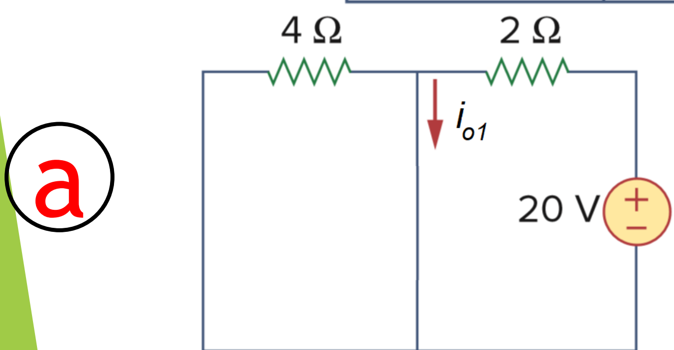
Answers: $i(t) = 4.472\sin(10t+3.43°)$ A, $v(t) = 8.944\sin(10t+93.43°)$ V

9.5 Impedance and Admittance (6)

<u>Example 9</u> Find i_o in the circuit using superposition.







$$\frac{V_{o2}-25}{4}+\frac{V_{o2}}{4j}+\frac{V_{o2}}{2}=0$$

$$V_{o2}-jV_{o2}+2V_{o2}=25 \quad \Rightarrow V_{o2}=\frac{25}{3-j}=\frac{25}{\sqrt{10}\angle -18.4^\circ}$$

$$V_{o2}=7.91\angle 18.4^\circ$$

$$I_{o2}=\frac{V_{o2}}{4j}=\frac{7.91\angle 18.4^\circ}{4\angle 90^\circ}=1.98\angle -71.6^\circ$$

$$\Rightarrow i_{o2}=1.98\cos(4t\text{-}71.6^\circ)$$

$$i_{o1}=\frac{20}{2}=10\text{ A}$$

$$i_o = i_{o1}+i_{o2}=10+1.98\cos(4t\text{-}71.6^\circ)\ \text{A}$$

9.6 Kirchhoff’s Laws in the Frequency Domain (1)

- Both KVL and KCL are hold in the phasor(frequency) domain.

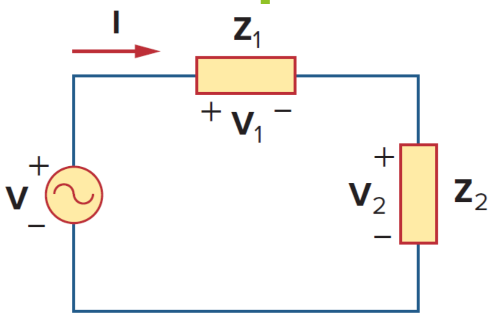
- Moreover, the variables to be handled are phasors, which are complex numbers.

9.7 Impedance Combinations (1)

- The following principles used for DC circuit analysis all apply to AC circuit.

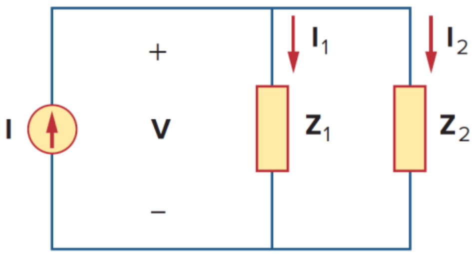
- For example:
  a. voltage division
  b. current division
  c. circuit reduction
  d. impedance equivalence
  e. Y-Δ transformation

9.7 Impedance Combinations (2)



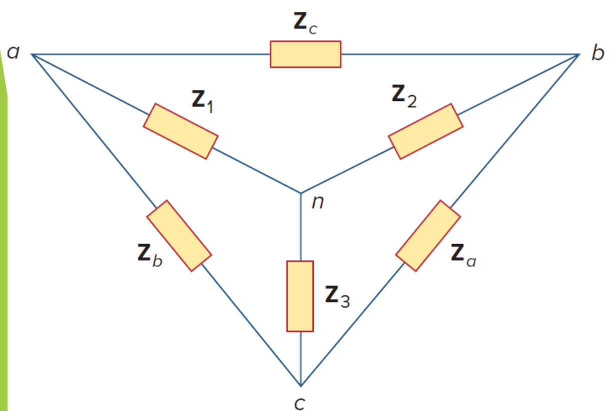
Voltage division.

$$I=\frac{V}{Z_1+Z_2},\quad V_1=\frac{Z_1}{Z_1+Z_2}V,\quad V_2=\frac{Z_2}{Z_1+Z_2}V$$



Current division.

$$V=I_1Z_1=I_2Z_2,\quad I_1=\frac{Z_2}{Z_1+Z_2}I,\quad I_2=\frac{Z_1}{Z_1+Z_2}I$$



Y and Δ networks.

Y-Δ *Conversion:*

$$\mathbf{Z}_a=\frac{\mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b=\frac{\mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c=\frac{\mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-*Y Conversion:*

$$\mathbf{Z}_1=\frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a+\mathbf{Z}_b+\mathbf{Z}_c}$$

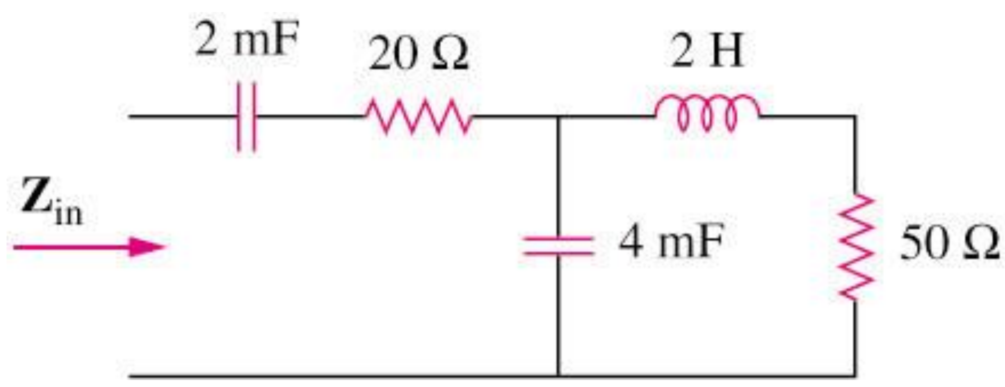
$$\mathbf{Z}_2=\frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a+\mathbf{Z}_b+\mathbf{Z}_c}$$

$$\mathbf{Z}_3=\frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a+\mathbf{Z}_b+\mathbf{Z}_c}$$

A delta or wye circuit is said to be **balanced** if it has equal impedances in all three branches.

9.7 Impedance Combinations (3)

Example 9 Determine the input impedance of the circuit in figure below at ω =10 rad/s.

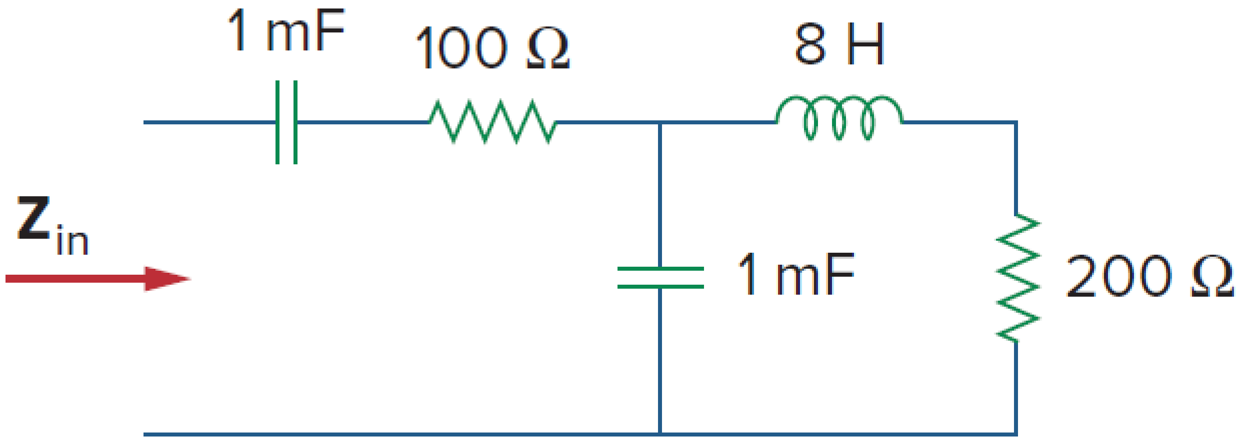


$$\left((50+20j)\|-25j\right)+20\text{-}50j = 32.38 - j73.76$$

Answer: $Z_{in} = (32.38 - j73.76)\ \Omega$

9.7 Impedance Combinations (4)

Example 10 Determine the input impedance of the circuit in figure below at ω =10 rad/s.



Answer: $Z_{in} = (149.52 - j195)\ \Omega$