Electrical Engineering 1

12026105

Chapter 8

Second-Order Circuits

Learning Objectives

By using the information and exercises in this chapter you will be able to:

1. Develop a better understanding of the solution of general-second order differential equations.
2. Learn how to determine initial and final values.
3. Understand the response in source-free series *RLC* circuits.
4. Understand the response in source-free parallel *RLC* circuits.
5. Understand the step response of series *RLC* circuits.
6. Understand the step response of parallel *RLC* circuits.
7. Understand general second-order circuits.
8. Understand general second-order circuits with op amps.

Second-Order Circuits Chapter 8

8.1 Examples of 2^{nd} order RCL circuit

8.2 The source-free series RLC circuit

8.3 The source-free parallel RLC circuit

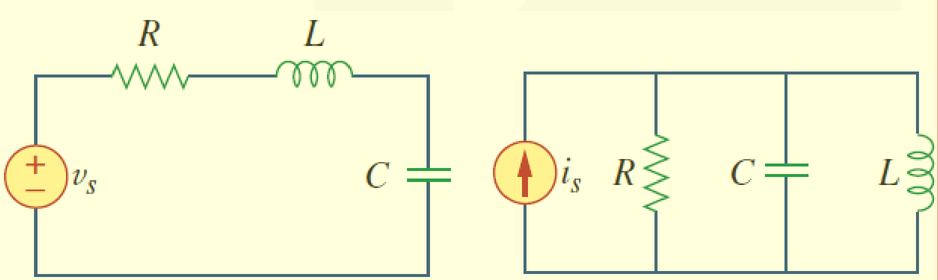
8.4 Step response of a series RLC circuit

8.5 Step response of a parallel

8.1 Second Order RLC circuits (1)

What is a 2nd order circuit?

A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

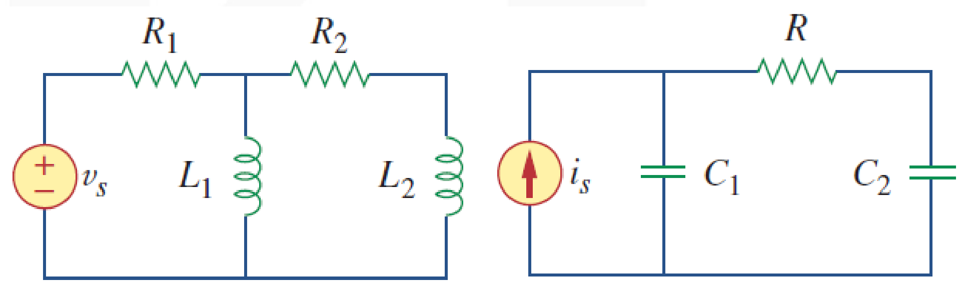


(a)

RLC Series

(b)

RLC Parallel



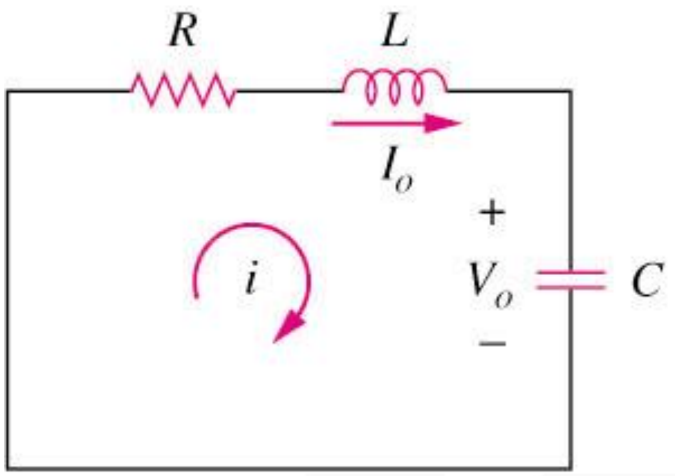
(c)

RL T-config

(d)

RC Pi-config

8.2 Source-Free Series RLC Circuits



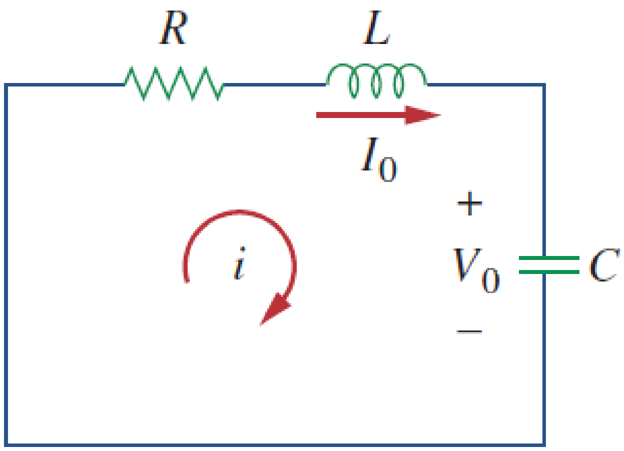
- The solution of the source-free series RLC circuit is called as the natural response of the circuit.
- The circuit is excited by the energy initially stored in the capacitor and inductor.

The 2nd order of expression

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

How to derive and how to solve?

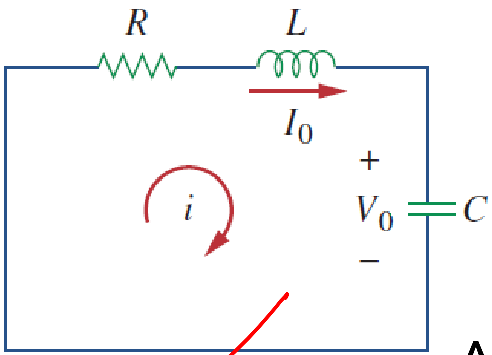
8.2 Source-Free Series RLC Circuits



$$For\ \ Capacitor: \quad v(0) = v(0^+) = v(0^-) = V_0$$

$$For\ \ Inductor: \quad i(0) = i(0^+) = i(0^-) = I_0$$

8.2 Source-Free Series RLC Circuits



<u>Initial Conditions</u>

1. $i(0) = I_0$

2. $$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$$

Apply KVL

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

At = $V_0 = L\frac{di}{dt}$

To solve such a 2nd order diff eq. We need 2 initial conditions, such as $i(0)$ and $\frac{di(0)}{dt}$ (from $v(0)$)

I_0 V_0

8.2 Source-Free Series RLC Circuits

We get the initial value of the derivative of *I* from equation after applying KVL; that is,

$$Ri(0) + L\frac{di(0)}{dt} + v(0) = 0$$

$$RI_0 + L\frac{di(0)}{dt} + V_0 = 0$$

V_0

2 Initial Conditions

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

$$i(0) = I_0$$

8.2 Source-Free Series RLC Circuits

In 1[st] order circuits suggests that the solution is $\boxed{I_0 e^{-t/\tau}}$. So we let

$i = Ae^{st}$ Where A and s have to be determined

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$$As^2 e^{st} + \frac{AR}{L} s e^{st} + \frac{Ae^{st}}{LC} = 0 \Rightarrow Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

a b c

$$\cancel{Ae^{st} = 0} \qquad 1 s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

8.2 Source-Free Series RLC Circuits

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s^2 + \alpha s + \omega_0^2 = 0$$

$i = Ae^{st} \rightarrow A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

8.2 Source-Free Series RLC Circuits

$z = a + bj \Rightarrow 3 + 2j$

There are 3 possible solutions for the following 2nd order differential equation:

$\sqrt{-1} = j$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$s_1 \rightarrow \alpha, \omega_0$

=> $$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = 0$$

Where $\alpha = \dfrac{R}{2L}$ and $\omega_0 = \sqrt{\dfrac{1}{LC}}$

General 2nd order Form

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

The types of solutions for i depend on the relative values of α and ω_0

8.2 Source-Free Series RLC Circuits

$i(t) = I_0 e^{-t/\tau}$

There are 3 possible solutions for the following 2nd order differential equation:

$$\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = 0$$

1. If $\alpha > \omega_o$, over-damped case

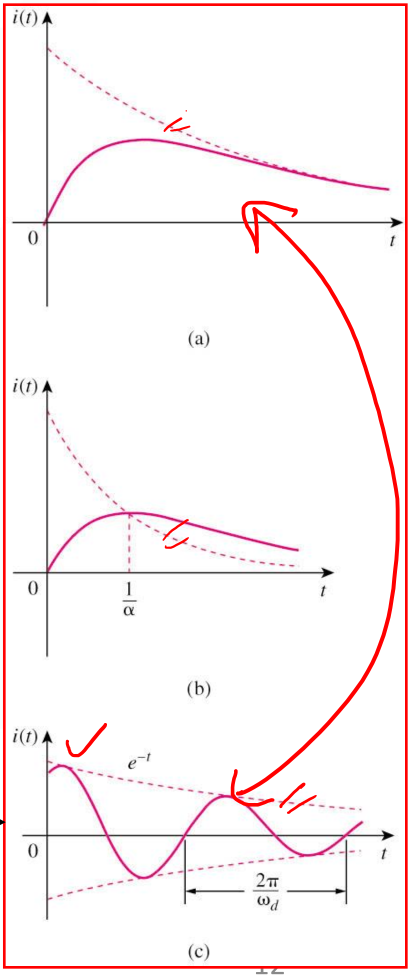
$$i(t) = Ae^{s_1 t} + Be^{s_2 t} \quad \text{where} \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

2. If $\alpha = \omega_o$, critical damped case

$$i(t) = (A + Bt)e^{-\alpha t} \quad \text{where} \quad s_{1,2} = -\alpha$$

3. If $\alpha < \omega_o$, under-damped case

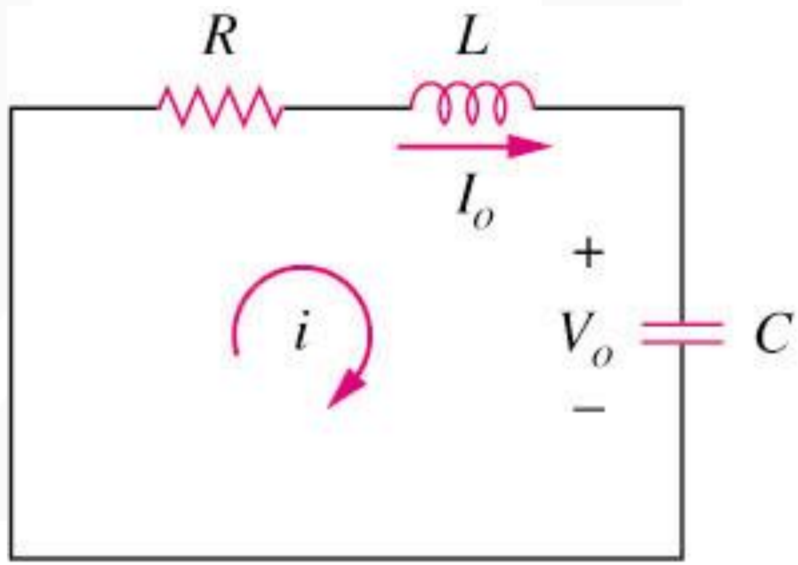
$$i(t) = e^{-\alpha t}(A\cos \omega_d t + B\sin \omega_d t) \quad \text{where} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$



8.2 Source-Free Series RLC Circuits

Example 1

If R = 10Ω, L = 5H, and C = 2mF in figure below, find α, ω_0, s_1 and s_2. What type of natural response will the circuit have?



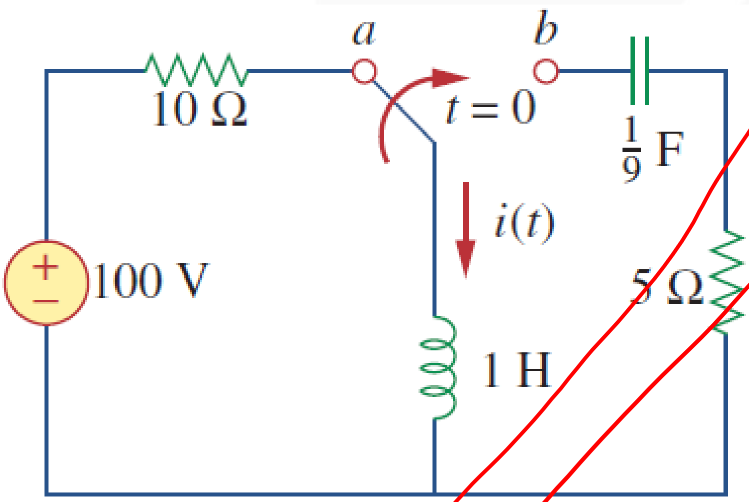
$$\alpha = \frac{R}{2L} = \frac{10}{2(5)} = 1$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{5(0.002)}} = 10$$

Answer: *underdamped*

8.2 Source-Free Series RLC Circuits

Example 2 The circuit shown below has reached steady state at t=0⁻. If the make-before-break switch moves to position b at t=0, calculate $i(t)$ for t > 0.



$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) = -\frac{1}{1}(50+0) = -50$$

$$i(0) = I_0 = 10\,\text{A}$$

$$\alpha = \frac{R}{2L} = \frac{5}{2(1)} = 2.5$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1(9)}{1}} = 3$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{3^2 - 2.5^2} = 1.6583$$

underdamped

$$i(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t) = e^{-2.5t}\big(10\cos(1.6583t) - 15.076\sin(1.6583t)\big)A$$

Answer: $i(t) = e^{-2.5t}\big(10\cos(1.6583t) - 15.076\sin(1.6583t)\big)A$

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) = -\frac{1}{1}(50+0) = -50$$

$$i(0) = I_0 = 10\,\text{A}$$

$$-25 = 1.6583\,B$$

$$B = \frac{-25}{1.6583} = -15.076$$

$$i(t) = e^{-2.5t}(A\cos\omega_d t + B\sin\omega_d t)$$

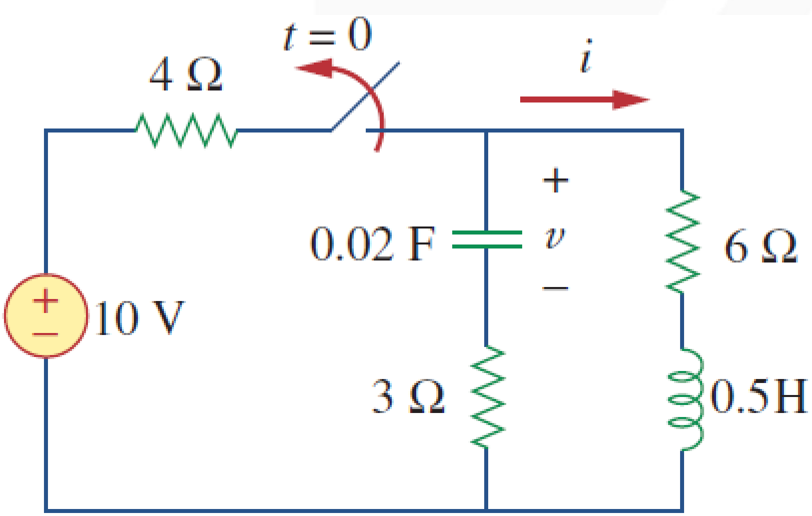
$$\frac{di(0)}{dt} = -50 = -2.5e^{-2.5t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-2.5t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)$$

$$= -2.5(10+0) + (0 + B\,1.6583)$$

$$i(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t) = e^{-2.5t}\big(10\cos(1.6583t) - 15.076\sin(1.6583t)\big)A$$

8.2 Source-Free Series RLC Circuits

Example 3 Find $i(t)$ in the circuit of Figure below. Assume that the circuit has reached steady state at t=0⁻.



$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) = -\frac{1}{0.5}(9-6) = -6$$

$$i(0) = I_0 = 1\text{A}$$

$$\alpha = \frac{R}{2L} = \frac{9}{2(0.5)} = 9$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.5(0.02)}} = 10$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^2 - 9^2} = 4.3589$$

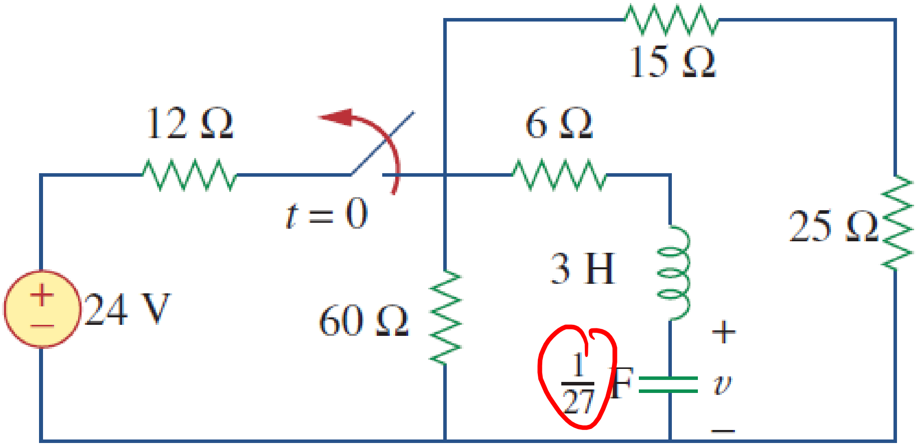
underdamped

$$i(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t) = e^{-9t}\left(1\cos(4.3589\,t) + 0.6882\sin(4.3589\,t)\right)A$$

Answer: $i(t) = e^{-9t}\left(\cos(4.3589\,t) + 0.6882\sin(4.3589\,t)\right)A$

8.2 Source-Free Series RLC Circuits

Example 4 Find $v(t)$ for t > 0



$$v(0) = V_0 = \left(\frac{24}{24+12}\right)24 = 16\text{V}$$

$$i(0) = I_0 = 0$$

$$\alpha = \frac{R}{2L} = \frac{30}{2(3)} = 5$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{27}{3}} = 3$$

overdamped

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -5 + \sqrt{5^2 - 3^2} = -1$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -5 - \sqrt{5^2 - 3^2} = -9$$

$A = -\frac{2}{3}$

$B = -A = +\frac{2}{3}$

$i = C\frac{dv}{dt}$

$i = Ae^{-t} - Ae^{-9t}$

$$v(t) = Ae^{-s_1 t} + Be^{-s_2 t} = Ae^{-t} + Be^{-9t}$$

$$v(0) = 16 = A + B$$

$$i(t) = C\frac{dv}{dt} = \frac{1}{27}\left(-Ae^{-t} - 9Be^{-9t}\right)$$

$$i(0) = 0 = \frac{1}{27}(-A - 9B) \rightarrow A = -9B$$

$$B = -2,\ A = 18$$

$V_0 = v\ \frac{1}{C}\int_{-\infty}^{0} i\,dt \rightarrow V$

$= \frac{1}{C}\left[-Ae^{-t} + \frac{A}{9}e^{-9t}\right]_0^{\infty}$

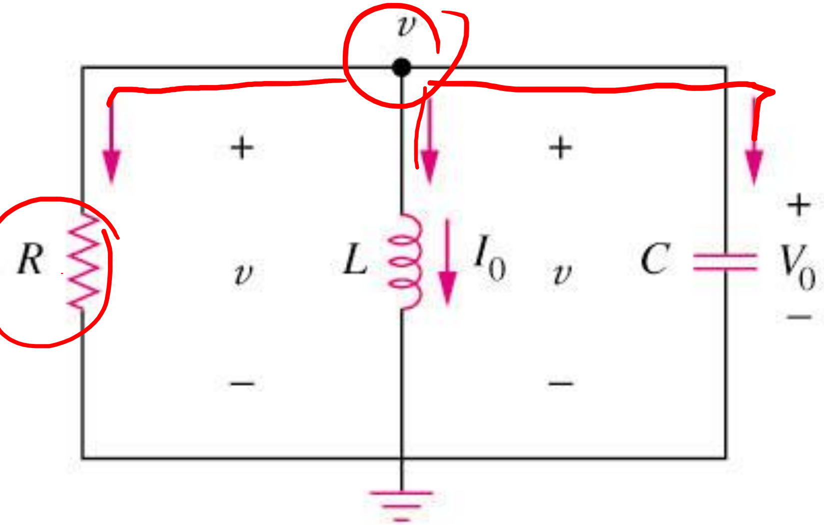
$= 27\left[-A + \frac{A}{9}\right]$

Answer:

$$i(t) = -\frac{2}{3}e^{-t} + \frac{2}{3}e^{-9t}$$

$$v(t) = 18e^{-t} - 2e^{-9t}$$

8.3 Source-Free Parallel RLC Circuits



Let $i(0)=I_0=\frac{1}{L}\int_{-\infty}^{0} v(t)dt$

$v(0) = V_0$, Apply KCL to the top node:

$I_R + I_L + I_C = 0$

$$\frac{v}{R}+\frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt}=0$$

$$\frac{1}{R}\frac{dv}{dt}+\frac{v}{L}+C\frac{d^2v}{dt^2}=0$$

Taking the derivative with respect to t and dividing by C

The 2nd order of expression

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

8.3 Source-Free Parallel RLC Circuits

There are 3 possible solutions for the following 2nd order differential equation:

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2 v = 0 \quad \text{where} \quad \alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

1. If $\alpha > \omega_o$, over-damped case

$$v(t) = A e^{s_1 t} + B e^{s_2 t}$$ where $$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

2. If $\alpha = \omega_o$, critical damped case

$$v(t) = (A + Bt)e^{-\alpha t}$$ where $$s_{1,2} = -\alpha$$

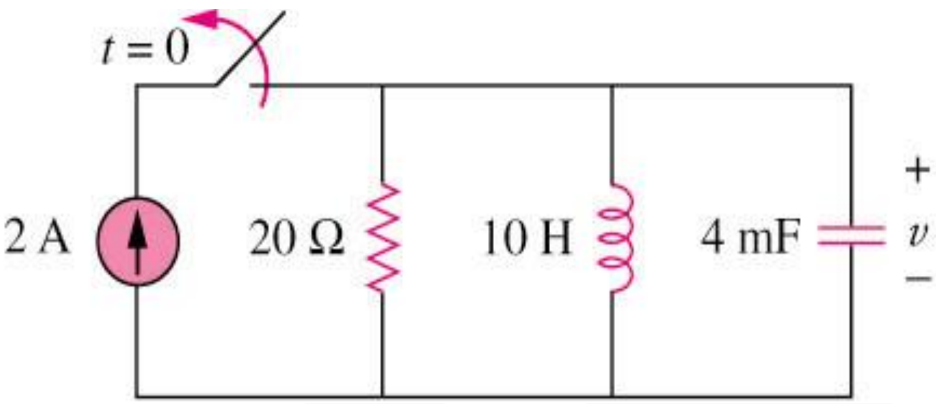
3. If $\alpha < \omega_o$, under-damped case

$$s_{1,2} = -\alpha \pm \sqrt{(-1)(\omega_0^2 - \alpha^2)}$$
$$= \alpha \pm \omega_d\sqrt{-1}$$
$$= \alpha \pm i\omega_d$$

$$v(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t)$$ where $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$
$$s_{1,2} = -\alpha \pm i\omega_d$$

8.3 Source-Free Parallel RLC Circuits

__Example 5__ Refer to the circuit shown below. Find $v(t)$ for $t > 0$.



$$v(0) = 0, i(0) = 2$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(20)(0.004)} = 6.25$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10(0.004)}} = 5$$

overdamped

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2} = -6.25 \pm \sqrt{6.25^2 - 5^2} = -6.25 \pm 3.75 = -2.5, -10$$

$$v(t) = A e^{s_1 t} + B e^{s_2 t} = Ae^{-2.5t} + Be^{-10t}$$

$$v(0) = 0 = A + B$$

$$v(t) = L\frac{di}{dt} \rightarrow i = \frac{1}{L}\int v(t)dt \rightarrow i(t) = \frac{1}{10}\int\left(Ae^{-2.5t} + Be^{-10t}\right)dt$$

$$i(0) = 2 = -\frac{A}{25} - \frac{B}{100}$$

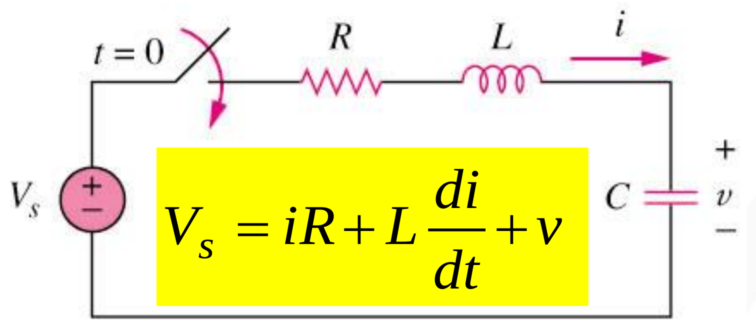
$$B = \frac{200}{3}, A = -\frac{200}{3}$$

Answer: $$v(t) = \frac{200}{3}\left(e^{-10t} - e^{-2.5t}\right)$$

$\rightarrow v(t) = Ae^{-2.5t} - Ae^{-10t}$ $\quad -\frac{200}{3} \quad +\frac{200}{3}$

8.4 Step-Response Series RLC Circuits

The step response is obtained by the sudden application of a dc source.



$$V_s = iR + L\frac{di}{dt} + v$$

$$i = C\frac{dv}{dt}$$

$$V_s = RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} + v$$

The 2nd order of expression

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

The above equation has the same form as the equation for source-free series RLC circuit.

- The same coefficients (important in determining the frequency parameters).
- Different circuit variable in the equation.

8.4 Step-Response Series RLC Circuits

The solution of the equation should have two components:

the transient response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$v(t) = v_t(t) + v_{ss}(t)$$

- The transient response v_t is the same as that for source-free case

$$v_t(t) = Ae^{s_1 t} + Be^{s_2 t}$$ (over-damped)

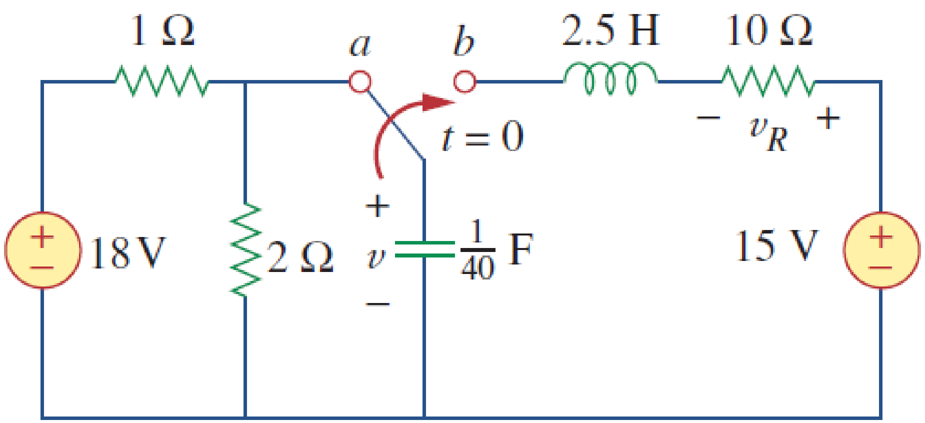
$$v_t(t) = (A + Bt)e^{-\alpha t}$$ (critically damped)

$$v_t(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t)$$ (under-damped)

- The steady-state response is the final value of v(t). $v_{ss}(t) = v(\infty)$
- A and B are obtained from the initial conditions : $v(0)$, $\dfrac{dv(0)}{dt}$

8.4 Step-Response Series RLC Circuits

<u>Example 6</u> Having been in position for a long time, the switch in the circuit below is moved to position b at t=0. Find v(t) and $v_R(t)$ for t > 0.



$$v(0)=\frac{2}{3}(18)=12V,\quad i(0)=0$$

$$v_{ss}(t)=15,\quad i(0)=C\frac{dv(0)}{dt}\rightarrow\frac{dv(0)}{dt}=0$$

$$\alpha=\frac{R}{2L}=\frac{10}{2(2.5)}=2,\ \omega_0=\sqrt{\frac{1}{LC}}=\sqrt{\frac{40}{2.5}}=4$$

$$\omega_d=\sqrt{\omega_0^2-\alpha^2}=2\sqrt{3}\quad under-damped$$

$$v(t)=v_{ss}(t)+v_t(t)=15+e^{-2t}(A\cos 2\sqrt{3}t+B\sin 2\sqrt{3}t)$$

$$v(0)=12=15+A\rightarrow A=-3$$

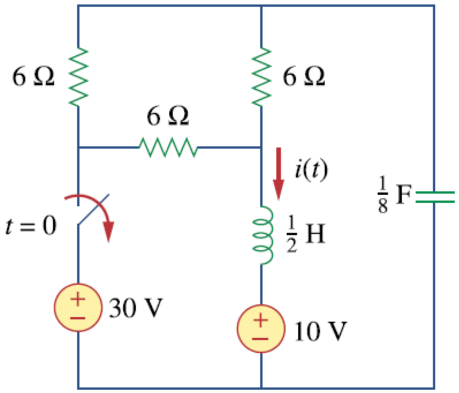
$$\frac{dv(t)}{dt}=\left(-2e^{-2t}(-3\cos 2\sqrt{3}t+B\sin 2\sqrt{3}t)\right)+\left(e^{-2t}(6\sqrt{3}\sin 2\sqrt{3}t+2\sqrt{3}B\cos 2\sqrt{3}t)\right)$$

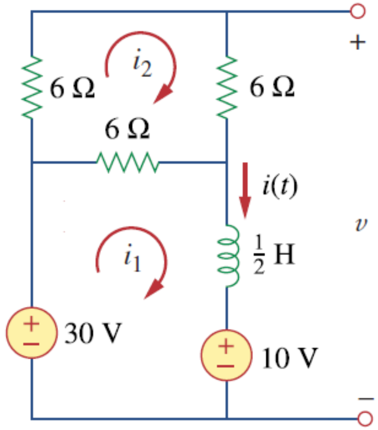
$$\frac{dv(0)}{dt}=0=6+2\sqrt{3}B\rightarrow B=-\sqrt{3}$$

Answer: $v(t)=15-e^{-2t}(3\cos(2\sqrt{3}t)+\sqrt{3}\sin(2\sqrt{3}t))V : v_R(t)=2\sqrt{3}e^{-2t}\sin(2\sqrt{3}t)V$

8.4 Step-Response Series RLC Circuits

Example 7 Find $i(t)$ in the circuit of Figure below.





$$i(0) = I_0 = 5\text{A}$$

$$i_1 = 5\,A,\ i_2 = \frac{5}{3}\,A,\ v(0) = 20\,V$$

$$R = 12//6 = 4\Omega,\ \alpha = \frac{R}{2L} = \frac{4(2)}{2} = 4$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{16} = 4,\ critically-damped$$

$$v(t) = v_{ss}(t) + v_t(t) = 10 + e^{-4t}(A + Bt)$$

$$v(0) = 20 = 10 + A \rightarrow A = 10$$

$$i(0) = -5 = \frac{e^{-4t}}{8}\left(-4(A + Bt) + B\right)$$

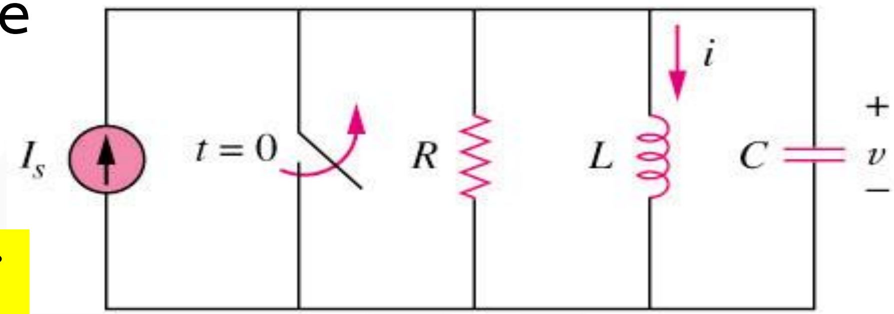
$$B = 0$$

$$i(t) = -C\frac{dv}{dt} = \frac{e^{-4t}}{8}(40) = 5e^{-4t}$$

Answer: $i(t) = 5e^{-4t}$

8.5 Step-Response Parallel RLC Circuits

The step response is obtained by the sudden application of a dc source.



$$C\frac{dv}{dt}+\frac{v}{R}+i=I_s$$

$$v = L\frac{di}{dt}$$

$$LC\frac{d^2i}{dt^2}+\frac{L}{R}\frac{di}{dt}+i=I_s$$

The 2nd order of expression

$$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}$$

It has the same form as the equation for source-free parallel RLC circuit.

- The same coefficients (important in determining the frequency parameters).
- Different circuit variable in the equation.

8.5 Step-Response Parallel RLC Circuits

The solution of the equation should have two components:

the transient response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$i(t) = i_t(t) + i_{ss}(t)$$

- The transient response i_t is the same as that for source-free case

$$i_t(t) = Ae^{s_1 t} + Be^{s_2 t}$$ (over-damped)

$$i_t(t) = (A + Bt)e^{-\alpha t}$$ (critical damped)

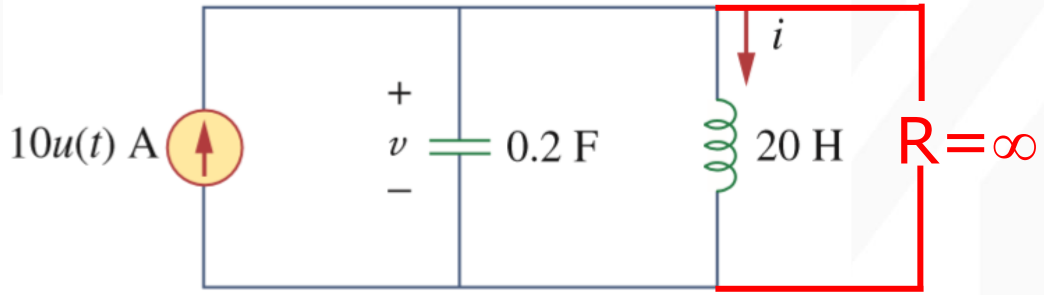
$$i_t(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t)$$ (under-damped)

- The steady-state response is the final value of i(t). $i_{ss}(t) = i(\infty) = I_s$
- The values of A and B are obtained from the initial conditions: $i(0)$, $\dfrac{di(0)}{dt}$

8.5 Step-Response Parallel RLC Circuits

Example 8

Find i(t) and v(t) for t > 0 in the circuit shown in circuit shown below:



$$\alpha = \frac{1}{2RC} = 0, \omega_0 = \frac{1}{\sqrt{LC}} = 0.5, \omega_0 = \omega_d$$

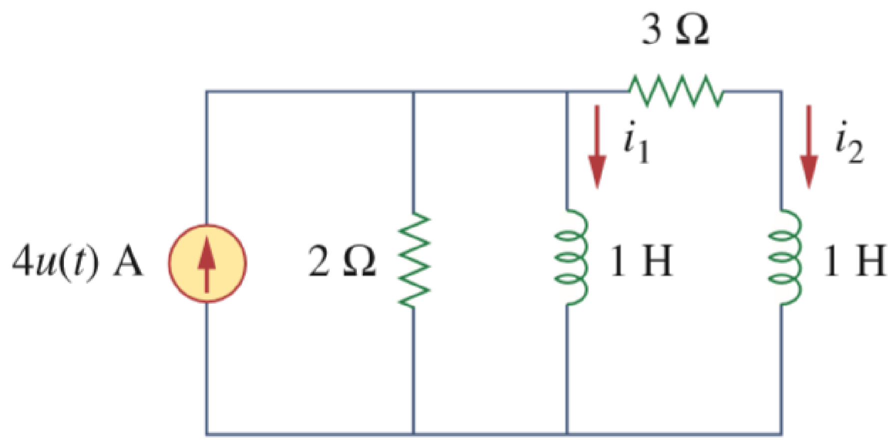
$$i_t(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t)$$

$$i(t) = A\cos\omega_d t + B\sin\omega_d t$$

Answer: $10(1 - \cos(0.25t))$ A, $50\sin(0.25t)$ V.

8.5 Step-Response Parallel RLC Circuits

Example 9 Find i_1 and i_2 for t > 0



At t = 0-, 4u(t) = 0 so that $i_1(0) = 0 = i_2(0)$ (1)

Applying nodal analysis,

$$4 = 0.5di_1/dt + i_1 + i_2 \quad (2)$$

Also, $i_2 = [1di_1/dt - 1di_2/dt]/3$ or $3i_2 = di_1/dt - di_2/dt$ (3)

Taking the derivative of (2), $0 = d^2i_1/dt^2 + 2di_1/dt + 2di_2/dt$ (4)

From (2) and (3), $di_2/dt = di_1/dt - 3i_2 = di_1/dt - 3(4 - i_1 - 0.5di_1/dt)$

$$= di_1/dt - 12 + 3i_1 + 1.5di_1/dt$$

8.5 Step-Response Parallel RLC Circuits

Substituting this into (4),

$$d^2i_1/dt^2 + 7di_1/dt + 6i_1 = 24 \text{ which gives } s^2 + 7s + 6 = 0 = (s + 1)(s + 6)$$

$$\text{Thus,} \quad i_1(t) = I_s + [Ae^{-t} + Be^{-6t}], \; 6I_s = 24 \text{ or } I_s = 4$$

$$i_1(t) = 4 + [Ae^{-t} + Be^{-6t}] \text{ and } i_1(0) = 4 + [A + B] \tag{5}$$

$$i_2 = 4 - i_1 - 0.5di_1/dt = i_1(t) = 4 + -4 - [Ae^{-t} + Be^{-6t}] - \frac{[-Ae^{-t} - 6Be^{-6t}]}{2}$$

$$= [-0.5Ae^{-t} + 2Be^{-6t}] \text{ and } i_2(0) = 0 = -0.5A + 2B \tag{6}$$

From (5) and (6), $A = -3.2$ and $B = -0.8$

$$i_1(t) = \underline{\mathbf{\{4 + [-3.2e^{-t} - 0.8e^{-6t}]\} \; A}}$$

$$i_2(t) = \underline{\mathbf{[1.6e^{-t} - 1.6e^{-6t}] \; A}}$$