Electrical Engineering 1

12026105

Chapter 7

First-Order Circuits

Learning Objectives

By using the information and exercises in this chapter you will be able to:

1. Understand solutions to unforced, first-order linear differential equations.
2. Comprehend singularity equations and know their importance in solving linear differential equations.
3. Understand the effect of unit step sources on first-order linear differential equations.
4. Explain how dependent sources and op amps influence simple first-order linear differential equations.
5. Use *PSpice* to solve simple transient circuits with an inductor or a capacitor.

First-Order Circuits

7.1 The Source-Free RC Circuit (วงจร RC ที่ไม่มีแหล่งจ่าย)

7.2 The Source-Free RL Circuit (วงจร RL ที่ไม่มีแหล่งจ่าย)

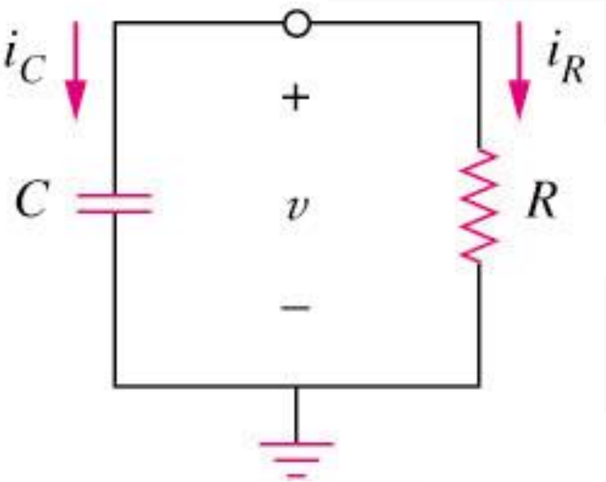
7.3 Unit-step Function

7.4 Step Response of an RC Circuit (วงจร RC ที่มีแหล่งจ่าย)

7.5 Step Response of an RL Circuit (วงจร RC ที่มีแหล่งจ่าย)

7.1 The Source-Free RC Circuit (1)

A first-order circuit is characterized by a first-order differential equation.



By KCL

$$i_R + i_C = 0 \rightarrow \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law

Capacitor law

$$\frac{dv}{v} = -\frac{dt}{RC}$$

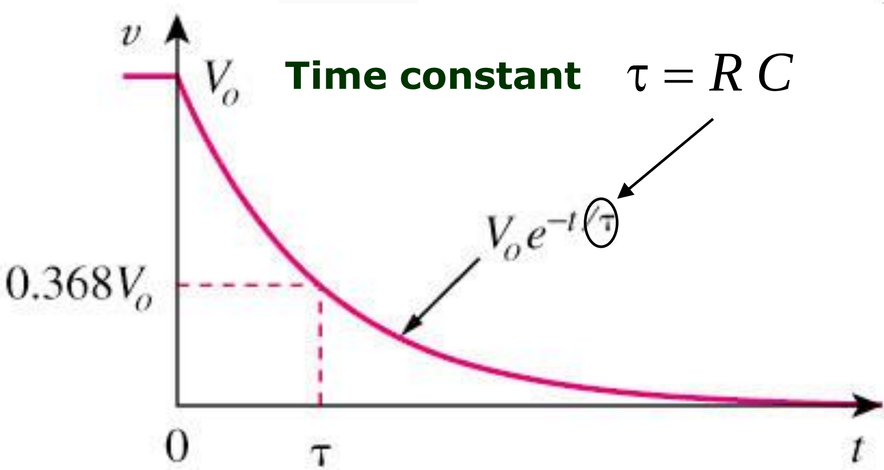
$$\ln(v) = -\frac{t}{RC} + Cont$$

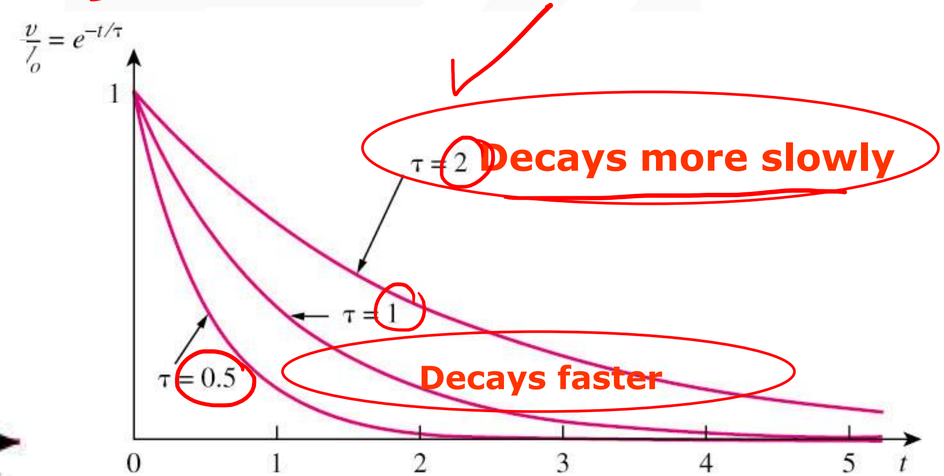
$$\boxed{v = e^{\left(-\frac{t}{RC}+Cont\right)} = V_o e^{\left(-\frac{t}{RC}\right)}}$$

- Apply Kirchhoff's laws to purely resistive circuit results in algebraic equations.
- Apply the laws to RC and RL circuits produces differential equations.

7.1 The Source-Free RC Circuit (2)

- The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.



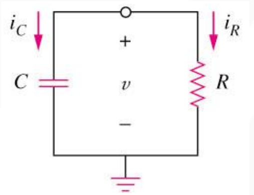


- The time constant τ of a circuit is the time required for the response to decay by a factor of 1/e or 36.8% of its initial value.

- v decays faster for small τ and slower for large τ.

7.1 The Source-Free RC Circuit (3)

The key to working with a source-free RC circuit is finding:



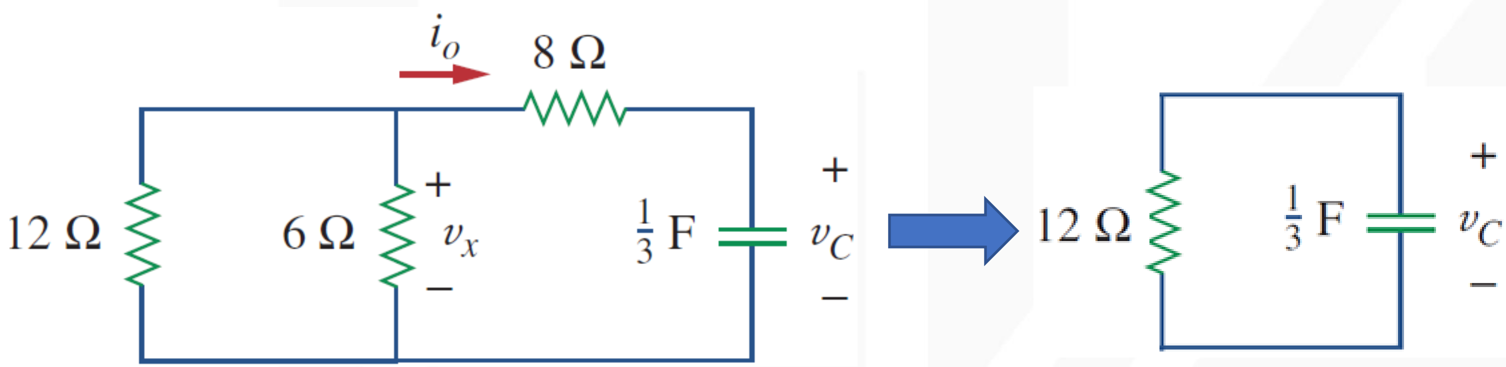
$$v(t) = V_0\, e^{-t/\tau}$$ where $$\tau = R\,C$$

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. τ = RC.

7.1 The Source-Free RC Circuit (4)

<u>Example 1</u>

Refer to the circuit below, determine v_C, v_x, and i_o for $t \geq 0$. Assume that $v_C(0)$ = 60 V.



$$v(t) = V_0\, e^{-t/\tau} = 60\, e^{-t/4}$$

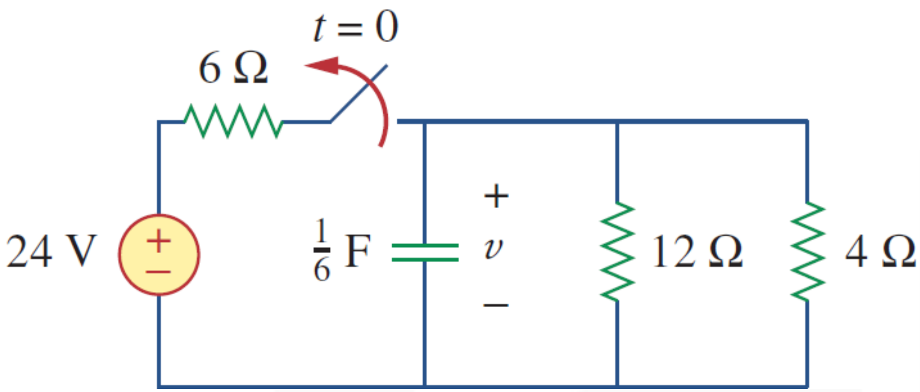
$$v_x(t) = \left(\frac{4}{4+8}\right) v_C(t) = 20\, e^{-t/4}$$

$$i_o(t) = C\frac{dv_C}{dt} = -5\, e^{-t/4}$$

<u>Answer</u>: $v_C = 60e^{-0.25t}$ V ; $v_x = 20e^{-0.25t}$ V; $i_o = -5e^{-0.25t}$ A

7.1 The Source-Free RC Circuit (5)

Example 2 The switch is opened at $t = 0$, find $v(t)$ for $t \geq 0$.



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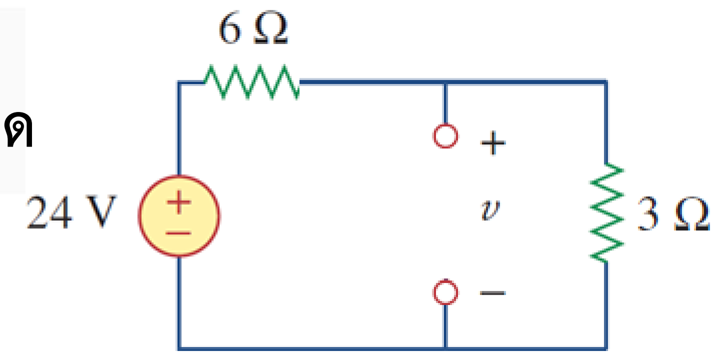

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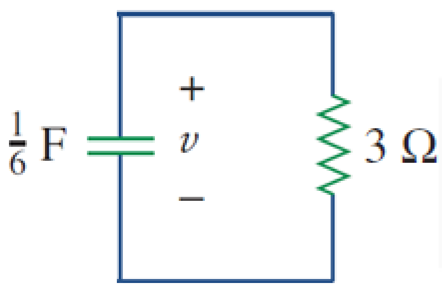
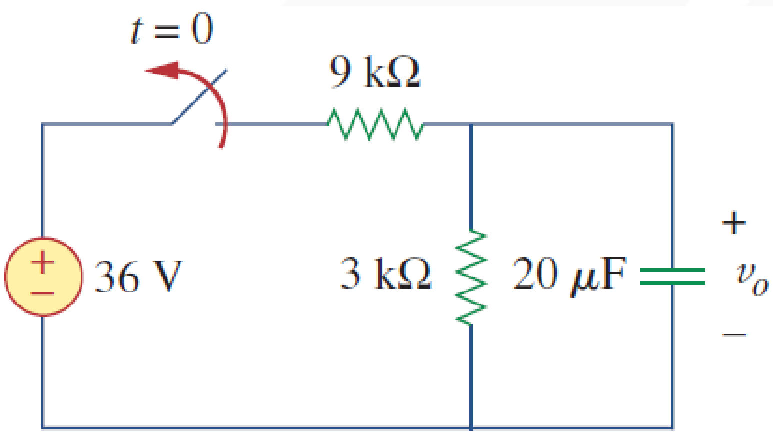


$$v(0) = V_0 = ? \rightarrow v(0) = \left(\frac{3}{3+6}\right) 24 = 8\,V$$

Answer: $v(t) = 8e^{-2t}$ V

7.1 The Source-Free RC Circuit (5)

Example 3 Find for $v_o(t)$ for t>0. Determine the time for the capacitor voltage to decay to one-third of its value at t=0



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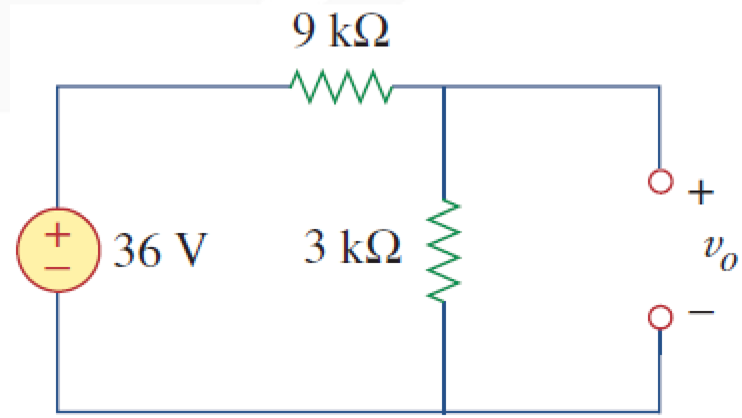

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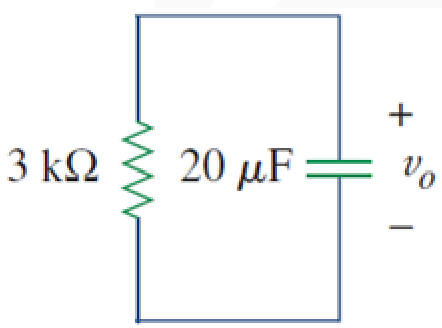


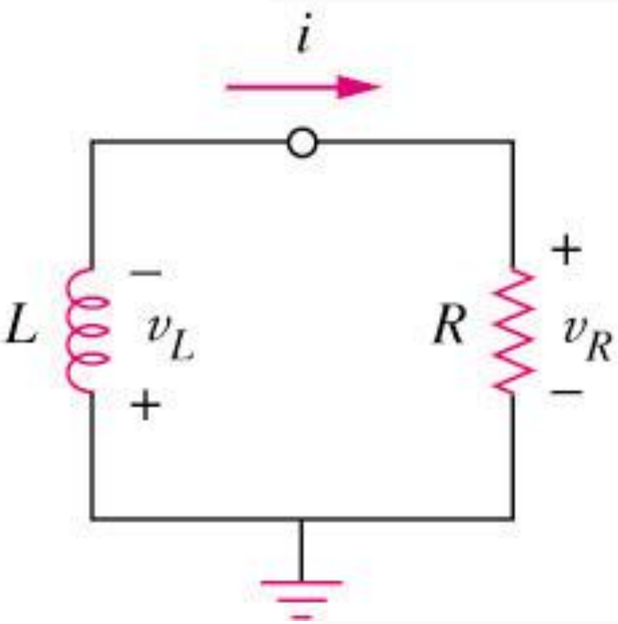
$$v(0) = V_0 = ? \rightarrow v(0) = \left(\frac{3}{3+9}\right)36 = 9\,V$$

$$9/3 = 3 = 9e^{-t/0.06}; t = -0.06 \times \ln(3/9) = 0.066\ s$$

Answer: $v(t) = 9e^{-t/0.06}$ V; t = 0.066 s

7.2 The Source-Free RL Circuit (1)

- A first-order RL circuit consists of L and R (or their equivalent)



By KVL

$$v_L + v_R = 0 \rightarrow L\frac{di}{dt} + iR = 0$$

Inductors law

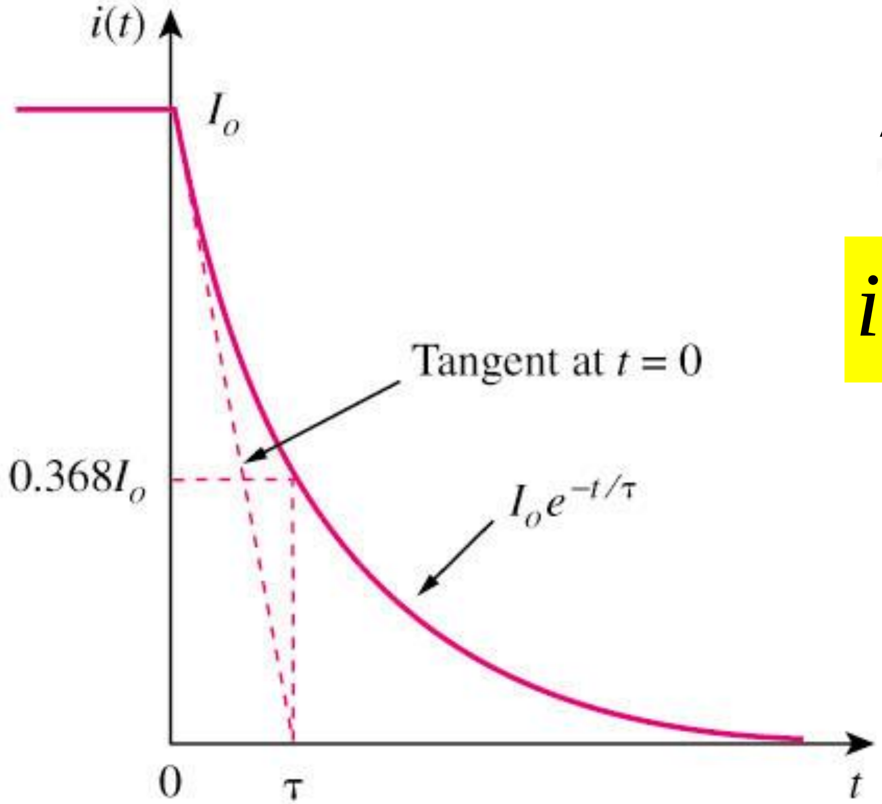
Ohms law

แก้สมการ

1st order dif eq

$$\rightarrow \frac{di}{i} = -\frac{R}{L}dt \rightarrow i(t) = I_0\, e^{-\left(\frac{R}{L}t\right)}$$

7.2 The Source-Free RL Circuit (2)



A general form representing RL

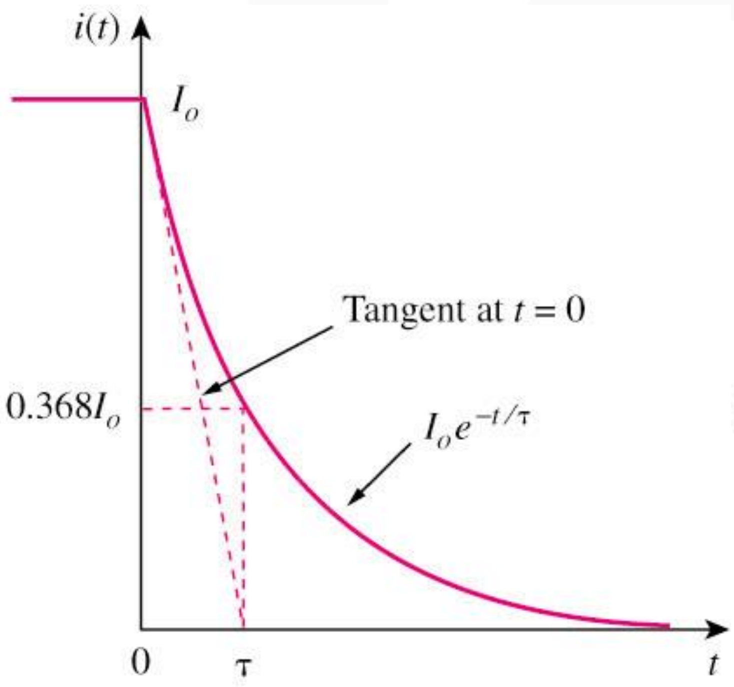
$$i(t) = I_0 \, e^{-t/\tau} \quad \text{where} \quad \tau = \frac{L}{R}$$

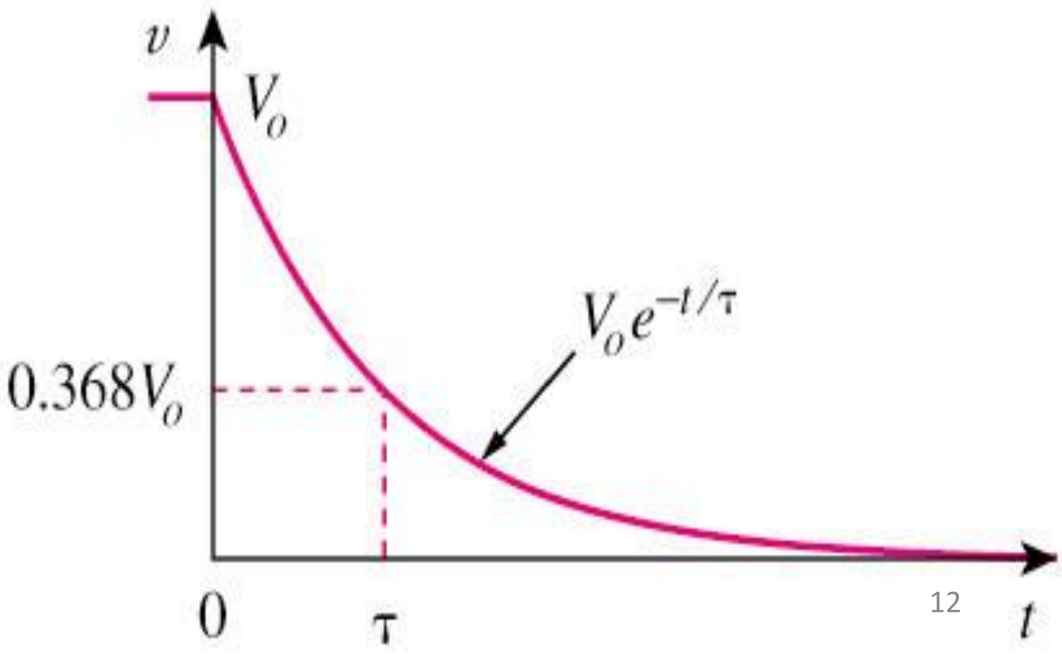
- The time constant τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- ***i*(*t*)** decays faster for small t and slower for large t.
- The general form is very similar to a RC source-free circuit.

7.2 The Source-Free RL Circuit (3)

Comparison between RL and RC circuit

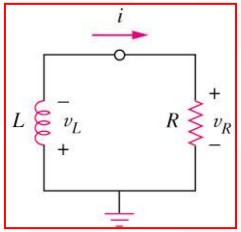
RL source-free circuit	RC source-free circuit
$i(t) = I_0\, e^{-t/\tau}$ where $\tau = \dfrac{L}{R}$	$v(t) = V_0\, e^{-t/\tau}$ where $\tau = RC$





7.2 The Source-Free RL Circuit (4)

The key to working with a source-free RL circuit is finding:

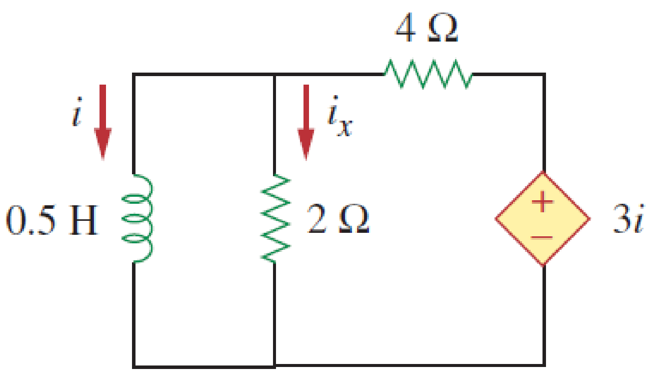


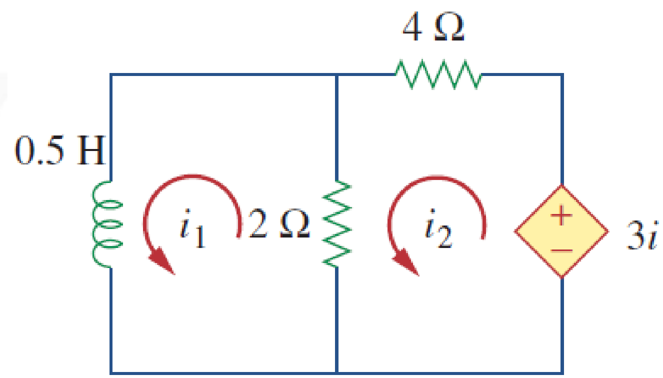
$$i(t) = I_0\, e^{-t/\tau}$$ where $$\tau = \frac{L}{R}$$

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau = L/R$.

7.2 The Source-Free RL Circuit (5)

Example 4   Assume that $i(0) = 10$ A. Find $i(t)$ and $i_x(t)$.





วิธีที่ 1 ใช้ KVL

loop 1, $\quad \dfrac{1}{2}\dfrac{di_1}{dt} + 2(i_1 - i_2) = 0$

loop 2, $\quad 6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \dfrac{5}{6}i_1$

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0 \qquad \text{Since } i_1 = i$$

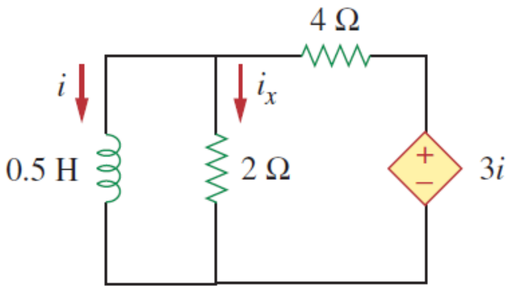
$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

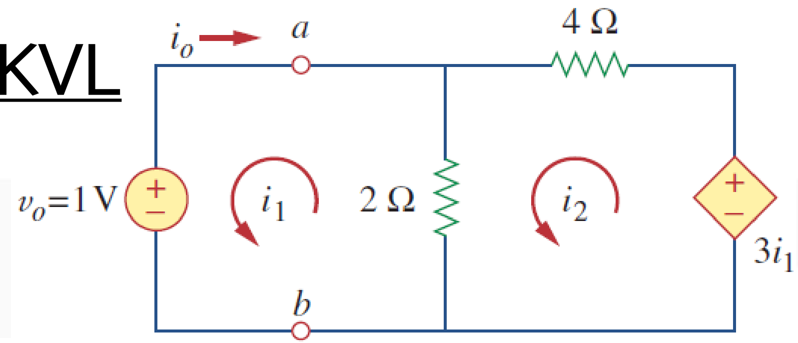
$$i_x(t) = i_2 - i_1 = -1.6667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Answer: $i(t) = 10\,e^{-2t/3}$ A; $i_x(t) = -1.6667\,e^{-2t/3}$ A

7.2 The Source-Free RL Circuit (5)

วิธีที่ 2 ใช้ R_{eq} และ KVL





$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1$$

$$i_1 = -3\text{ A}, \qquad i_o = -i_1 = 3\text{ A}$$

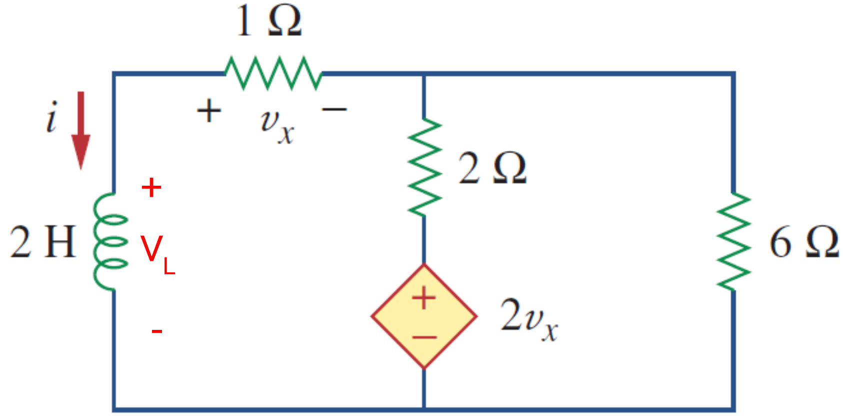
$$R_{\text{eq}} = R_{\text{Th}} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega \qquad \tau = \frac{L}{R_{\text{eq}}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

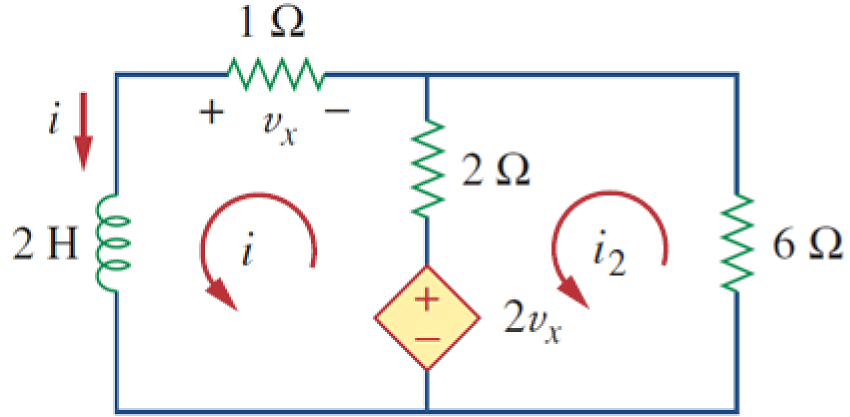
$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t}\text{ A}, \qquad t > 0$$

$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t}\text{ A}, \qquad t > 0$$

7.2 The Source-Free RL Circuit (5)

Example 5 Find i and v_x. Assume that $i(0)$ = 12 A.





$$\text{loop 1,}\quad 2\frac{di}{dt}+2(i-i_2)-3v_x=0$$

$$v_x=-i$$

$$\text{loop 2,}\quad 8i_2-2i+2v_x=0\quad ,8i_2+4v_x=0\quad ,i_2=-0.5\,v_x=0.5i$$

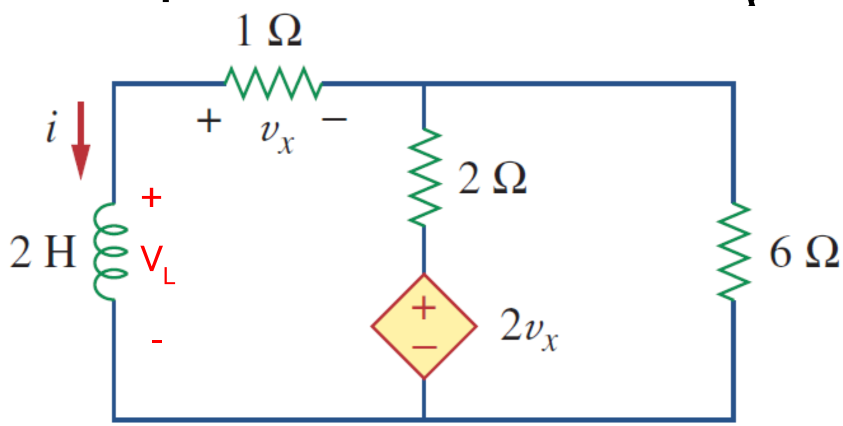
$$2\frac{di}{dt}+2(i-0.5i)+3i=0\ \Rightarrow\ 2\frac{di}{dt}+4i=0\ \Rightarrow\ \frac{di}{dt}+2i=0$$

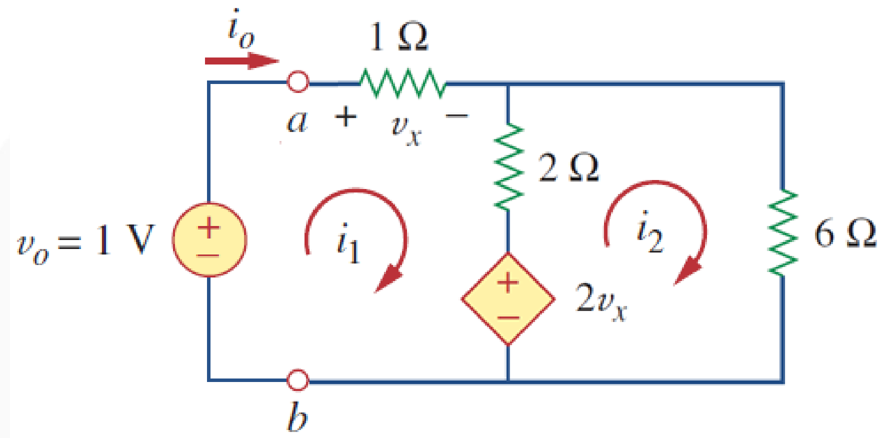
$$\frac{di}{i}=-2dt\ \Rightarrow\ \ln(i)=-2t\ \Rightarrow\ i=I_o e^{-2t}$$

Answer: $i(t)=12\,e^{-2t}$ A; $v_x(t)=-i(t)=-12\,e^{-2t}$ V

7.2 The Source-Free RL Circuit (5)

Example 5   Find i and v_x. Assume that $i(0)$ = 12 A.





$$\text{loop 1,} \quad 3i_1 - 2i_2 + 2i_1 = 5i_1 - 2i_2 = 1$$

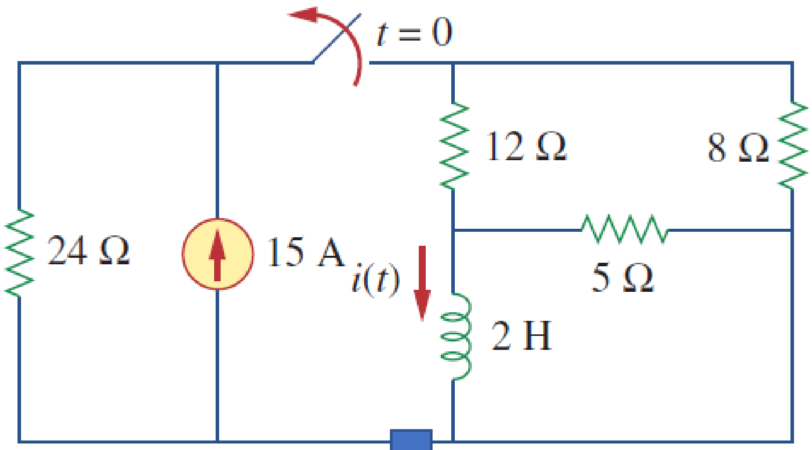
$$\text{loop 2,} \quad 2i_1 = -2i_1 + 8i_2 \quad i_1 = 2i_2$$

$$i_2 = \frac{1}{8}, i_1 = \frac{1}{4}$$

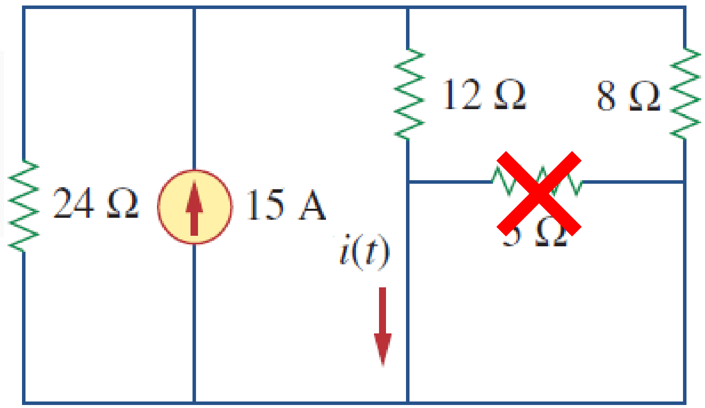
$$R_{eq} = \frac{v_o}{i_o} = 4\Omega, \quad \tau = \frac{L}{R_{eq}} = \frac{2}{4} = \frac{1}{2}$$

Answer: $i(t) = 12\,e^{-2t}$ A; $v_x(t) = -i(t) = -12\,e^{-2t}$ V

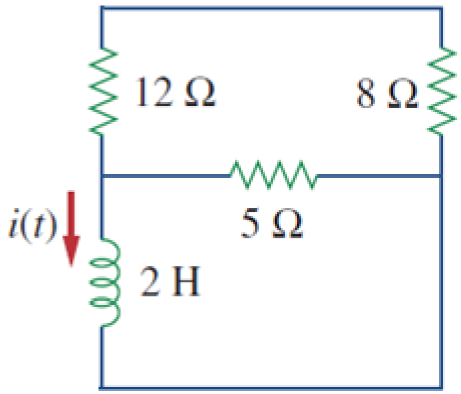
<u>Example 6</u> For the circuit, find $\boldsymbol{i\,(t)}$ for $\boldsymbol{t} > 0$.



t<0



t>0



$$i(0) = I_0 = ? \rightarrow i(0) = \left(\frac{24//8}{(24//8)+12}\right)15 = 5\text{ A}$$

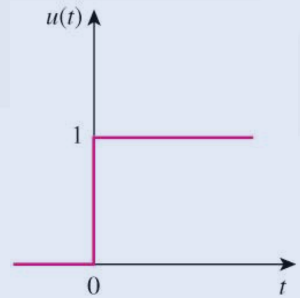
$$R_{eq} = (12+8)//5 = 4$$

<u>Answer</u>: $\boldsymbol{i(t) = 5e^{-2t}}$ A , t >0

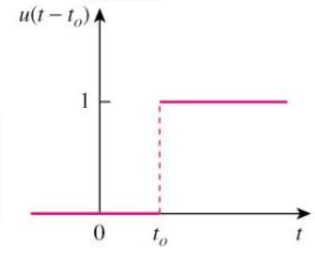
7.3 Unit-Step Function (1)

- The *unit step function* $\boldsymbol{u(t)}$ is 0 for negative values of $\boldsymbol{t}$ and 1 for positive values of $\boldsymbol{t}$.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

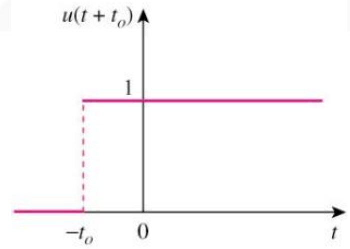


$$u(t - t_o) = \begin{cases} 0, & t < t_o \\ 1, & t > t_o \end{cases}$$



(a)

$$u(t + t_o) = \begin{cases} 0, & t < -t_o \\ 1, & t > -t_o \end{cases}$$

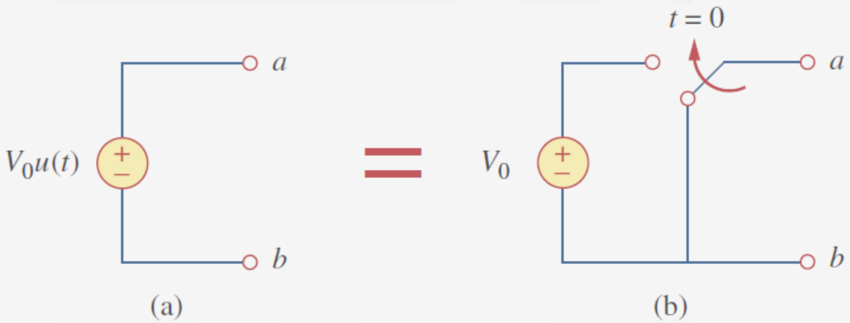


(b)

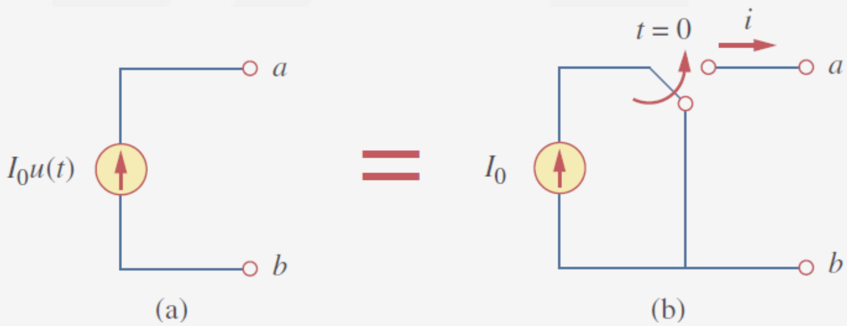
7.3 Unit-Step Function (2)

Represent an abrupt change for:

1. voltage source:

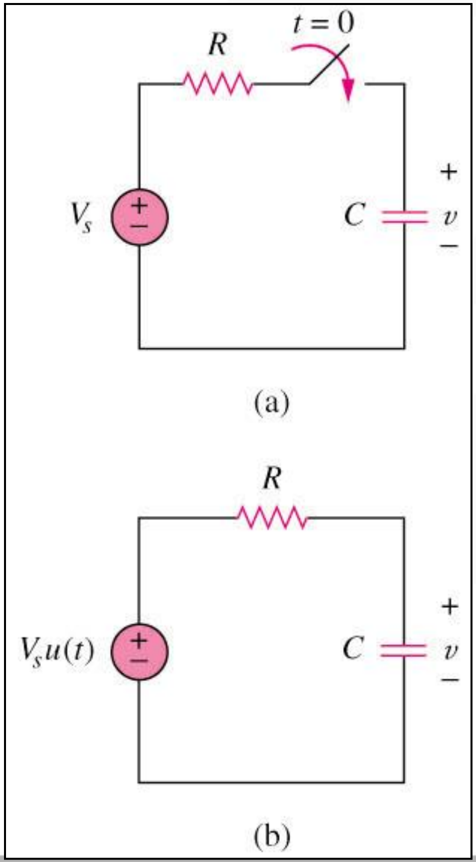


2. current source:



7.4 The Step-Response of a RC Circuit (1)

- The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- Initial condition: $v(0^-) = v(0^+) = V_0$

- Applying KCL, $C\dfrac{dv}{dt} + \dfrac{v - V_s u(t)}{R} = 0$

or

$$\frac{dv}{dt} = -\frac{v - V_s u(t)}{RC}$$

- Where $u(t)$ is the unit-step function

$$\int \frac{dv}{v - V_s} = -\int \frac{dt}{RC}$$

$$\ln(v - V_s)\Big]_{V_0}^{v(t)} = -\frac{t}{RC}\Big]_0^t$$

7.4 The Step-Response of a RC Circuit (1)

$$\ln(v-V_s)\Big]_{V_0}^{v(t)} = -\frac{t}{RC}\Big]_0^t$$

$$\ln(v(t)-V_s)-\ln(V_0-V_s) = -\frac{t}{RC}$$

$$\ln\left(\frac{v(t)-V_s}{V_0-V_s}\right) = -\frac{t}{RC}$$

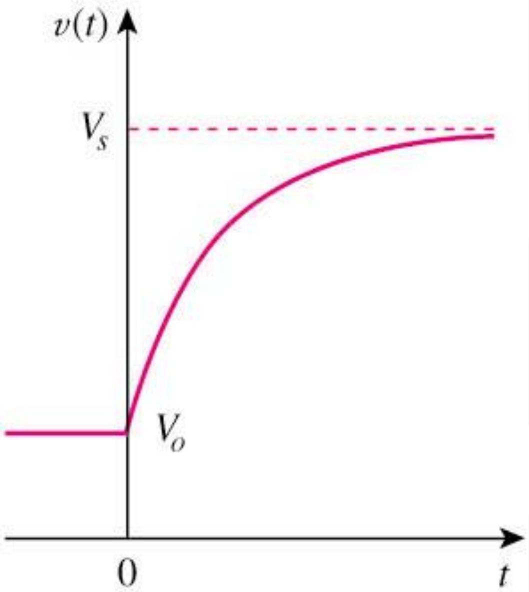
$$\frac{v(t)-V_s}{V_0-V_s} = e^{-\frac{t}{RC}}$$

$$v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{RC}}$$

$$v(t) = v(\infty) + \big(v(0) - v(\infty)\big)e^{-\frac{t}{\tau}}$$

7.4 The Step-Response of a RC Circuit (2)

- Integrating both sides and considering the initial conditions, the solution of the equation is:



Steady-state Response (permanent part)

Transient Response (temporary part)

$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}$$

Final value at t $\rightarrow$ ∞

Initial value at t = 0

Source-free Response

Complete Response = Natural response + Forced Response

(stored energy) (independent source)

(เกิดจาก no source) (เกิดจาก source)

$$V_0 e^{-t/\tau} + V_s(1 - e^{-t/\tau})$$

7.4 The Step-Response of a RC Circuit (3)

3 steps to find out the step response of an RC circuit:

1. Initial capacitor voltage $v(0)$.
2. Final capacitor voltage $v(\infty)$ — DC voltage across C.
3. Time constant τ.

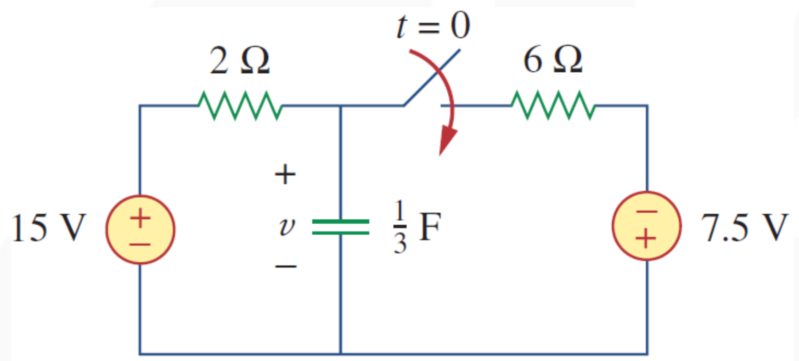
$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

7.4 The Step-Response of a RC Circuit (4)

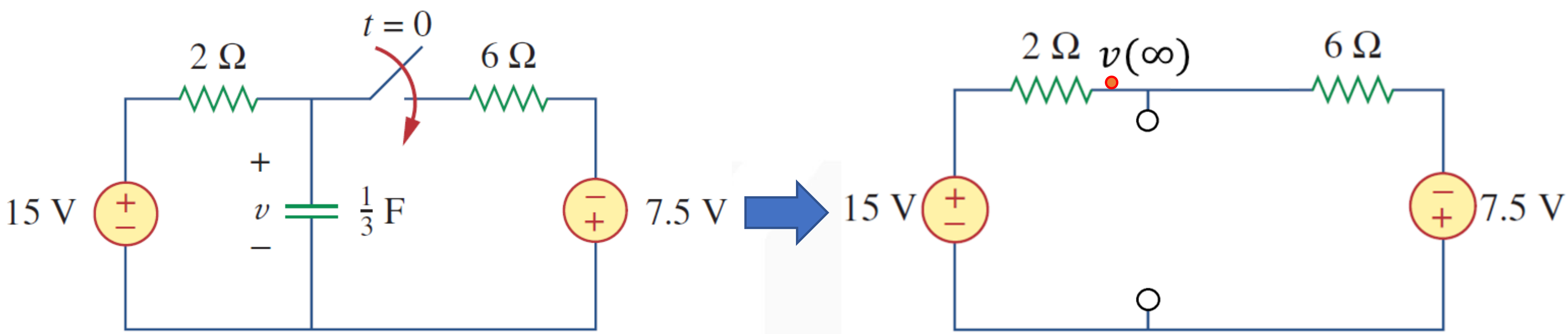
Example 7

Find $v(t)$ for t>0 in the circuit in below. Assume the switch has been open for a long time and is closed at t=0. Calculate $v(t)$ at t=0.5.



Answer: $v(t) = (5.625e^{-2t} + 9.375)V$ เมื่อ t>0, $v\,(0.5) = 11.44\ V$

7.4 The Step-Response of a RC Circuit (5)



$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$v(0^-) = v(0^+) = 15$ โวลต์

$v(\infty) = v(0^+) = 9.375$ โวลต์

$$i = \left(\frac{22.5}{6+2}\right), \qquad v(\infty) = 15 - \left(\frac{22.5}{6+2}\right)2 = 9.375 \text{ โวลต์}$$

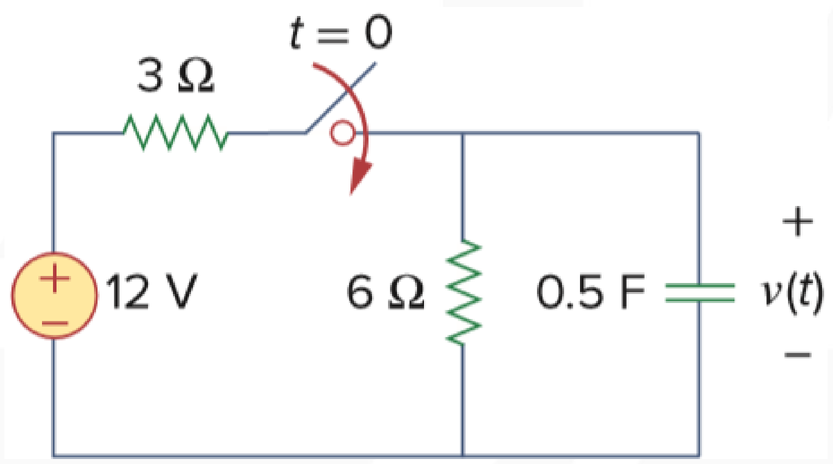
$$\mathrm{R} = 2 \,//\, 6 = \frac{3}{2}\Omega, \qquad \mathrm{C} = \frac{1}{3}, \qquad RC = \frac{1}{2} \qquad -\frac{t}{\tau} = -\frac{t}{RC} = -2t$$

<u>Answer</u>: $v(t) = (5.625e^{-2t} + 9.375)V$ เมื่อ t>0, $v(0.5) = 11.44\ V$

7.4 The Step-Response of a RC Circuit (6)

Example 8

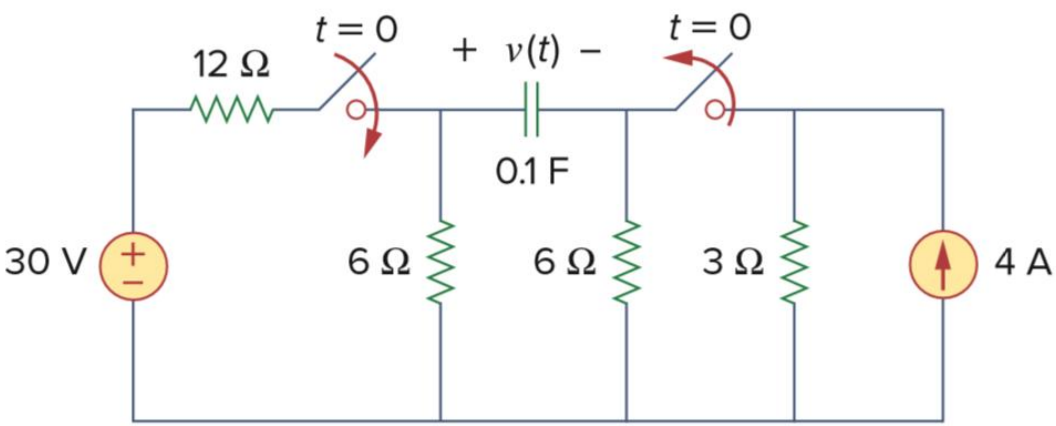
Find $v(t)$ for t>0 in the circuit in below. Assume the switch has been open for a long time and is closed at t=0.

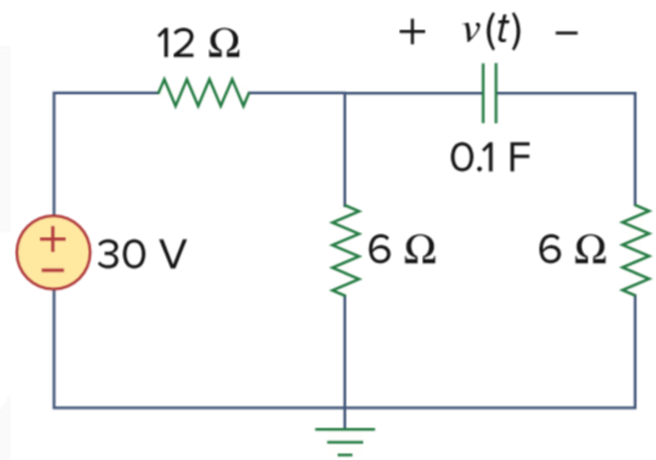


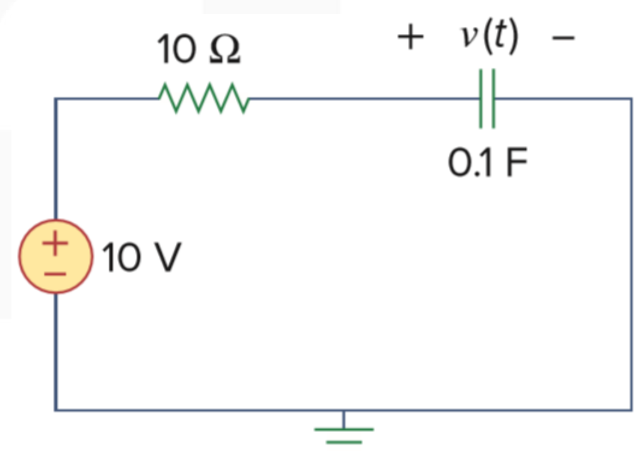
Answer : $v(t) = 8(1 - e^{-t})$ V, $t > 0$.

7.4 The Step-Response of a RC Circuit (7)

Example 9 Find $v(t)$ for t>0 in the circuit in below.



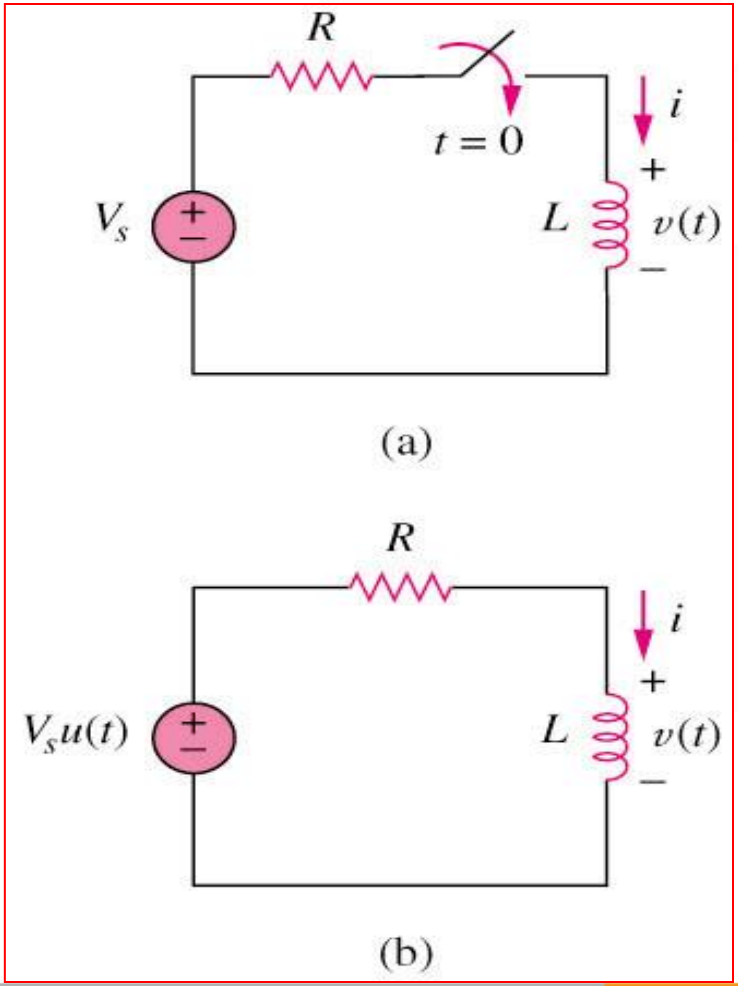




Answer : $v(t) = 10 - 18e^{-t}$ V

7.5 The Step-response of a RL Circuit (1)

- The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- Initial current $i(0^-) = i(0^+) = I_0$
- Final inductor current

$$i(\infty) = V_s / R$$

- Time constant $\tau = L/R$

$$i(t) = \frac{V_s}{R} + \left(I_o - \frac{V_s}{R}\right)e^{-\frac{t}{\tau}}$$

$$i(t) = i(\infty) + \left(i(0^+) - i(\infty)\right)e^{-\frac{t}{\tau}}$$

7.5 The Step-Response of a RL Circuit (2)

3 steps to find out the step response of an RL circuit:

1. Initial inductor current $i(0)$ at $t = 0^+$.
2. Final inductor current $i(\infty)$.
3. Time constant τ.

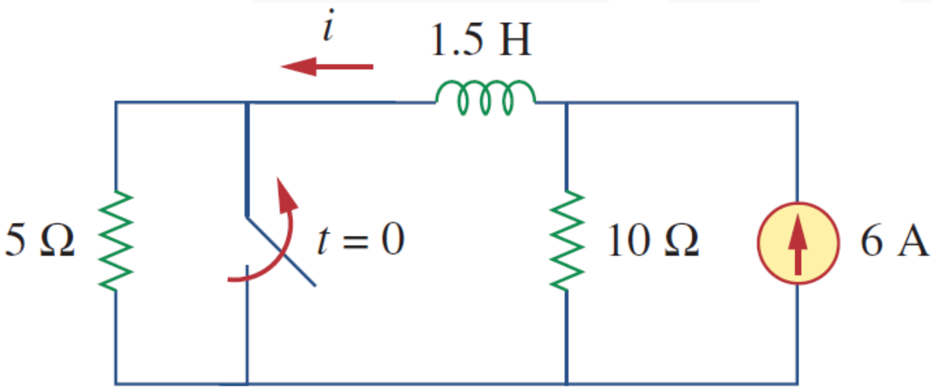
$$i(t) = i(\infty) + [i(0+) - i(\infty)]e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

7.5 The Step-Response of a RL Circuit (3)

Example 10

The switch in the circuit shown below has been closed for a long time. It opens at t = 0. Find i(t) for t > 0.



$$i(0) = \left(\frac{10}{10+0}\right)6 = 6\text{ A}$$

$$i(\infty) = \left(\frac{10}{5+10}\right)6 = 4\text{ A}$$

$$R_{eq} = 5+10 = 15\Omega \quad ;\tau = \left(\frac{L}{R_{eq}}\right) = \frac{1.5}{15} = \frac{1}{10}$$

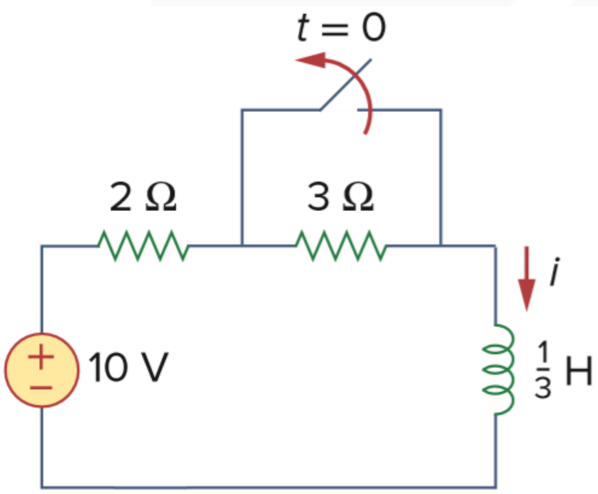
$$i(t) = i(\infty) + [i(0+) - i(\infty)]\,e^{-t/\tau} = 4 + (6-4)e^{-10t} = 4 + 2e^{-10t}$$

Answer: $i(t) = 4 + 2e^{-10t}$

7.5 The Step-Response of a RL Circuit (4)

Example 11

The switch in the circuit shown below has been closed for a long time. It opens at t = 0. Find i(t) for t > 0.



$$i(0^-) = \frac{10}{2} = 5 \text{ A} \qquad i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

$$R_{\text{Th}} = 2 + 3 = 5\ \Omega \qquad \tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

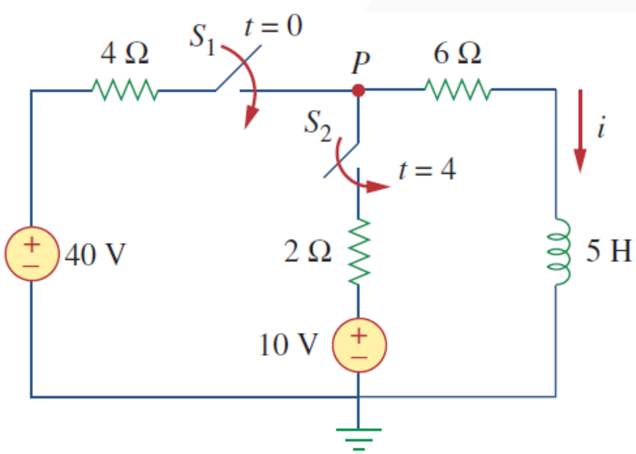
$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

$$= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A},$$

Answer: $i(t) = 2 + 3e^{-15t}$ A

7.5 The Step-Response of a RL Circuit (5)

Example 12 At $t=0$ switch 1 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for t>0. Calculate i for $t=2$ s and $t=5$ s.



three time intervals $t<0,\ 0\le t\le 4,\ t>4$

$$t<0: \quad i(0^-)=i(0^+)=0$$

$$0\le t\le 4: i(\infty)=\frac{40}{4+6}=4,\ R_{eq}=10\Omega \qquad \tau=\frac{L}{R_{eq}}=\frac{1}{2}s$$

$$i(t) = i(\infty)+[i(0+)-i(\infty)]e^{-t/\tau}=4+(0-4)e^{-2t}=4\left(1-e^{-2t}\right)$$

$$t>4: \quad i(4)=i(4^-)=4\left(1-e^{-8}\right)\approx 4A$$

$$KCL \text{ at node P}: \quad \frac{40-v}{4}+\frac{10-v}{2}=\frac{v}{6}\rightarrow v=\frac{180}{11}$$

$$i(\infty)=\frac{v}{6}=\frac{30}{11}=2.727A,\ R_{eq}=(4//2)+6=\frac{22}{3}\Omega \qquad \tau=\frac{L}{R_{eq}}=\frac{15}{22}s$$

$$i(t) = i(\infty)+[i(4)-i(\infty)]e^{-t/\tau}=2.727+(4-2.727)e^{-(t-4)/\tau}=2.727+1.273e^{-1.4667(t-4)}$$

Answer: $i(2)=4\left(1-e^{-4}\right)=3.93A, i(5)=2.727+1.273e^{-1.4667}=3.02A$